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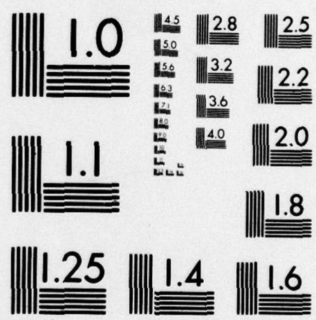
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CONTINGENT DERIVATIVES OF SET-VALUED MAPS AND EXISTENCE OF SOLUTIONS TO NONLINEAR INCLUSIONS AND DIFFERENTIAL INCLUSIONS

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ABSTRACT

We use the Bouligand contingent cone to a subset K of a Hilbert space at $x \in K$ for defining contingent derivatives of a set-valued map, whose graphs are the contingent cones to the graph of this map, as well as the upper contingent derivatives of a real valued function. We develop a calculus of these concepts and show how they are involved in optimization problems and in solving equations $f(x) = 0$ and/or inclusions $0 \in F(x)$. They also play a fundamental role for generalizing the Nagumo theorem on flow invariance and for generalizing the concept of Liapunov functions for differential equations and/or differential inclusions.

AMS(MOS) Subject Classification: 47H10, 47H15, 47H99, 49B30, 49B99

Key Words: contingent cones, contingent derivatives, nonlinear equations, set-valued map

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SIGNIFICANCE AND EXPLANATION

In recent years, the concept of multifunction (or set-valued mapping) has proved increasingly useful. A multifunction F is a mapping $x \to F(x)$ such that, for each x, $F(x)$ is a set (rather than a point, as would be the case if F were a function in the usual sense). In this paper we propose a definition of derivatives for such a multifunction, and we go on to develop some of its properties and applications. The key to the definition is to consider the graph of F; i.e. the set of all points (x,y) such that $y \in F(x)$. We take the tangent cone to this graph at a particular point (x_0,y_0), and we define the derivative of F at (x_0,y_0) to be the multifunction whose graph is this cone. (One discerns here an analogue of the familiar property of differentiable functions: the tangent line to the graph of F at the point $(x_0,F(x_0))$ has slope $F'(x)$.) The question arises as to what tangent cone to choose. If the graph of F is convex, there is no ambiguity. If not, we have the choice between the Bouligand contingent cone, which is large but not necessarily convex, and the Clarke tangent cone, which is always convex, but "too small" in some instances. We choose in this paper the Bouligand tangent cone because it appears naturally in the following contexts:

- studying necessary conditions in nonsmooth optimization problems and sensitivity analysis

-giving sufficient conditions for the existence of solutions to inclusions $0 \in F(x)$

- giving necessary and sufficient solutions for trajectories of a differential inclusion to remain in a given subset (invariance)

- giving necessary and sufficient conditions for trajectories of a differential inclusion to satisfy conditions of the form: $t \to V(x(t))$ is non-decreasing (stability).

CONTINGENT DERIVATIVES OF SET-VALUED MAPS
AND EXISTENCE OF SOLUTIONS
TO NONLINEAR INCLUSIONS AND DIFFERENTIAL INCLUSIONS

Jean Pierre Aubin

Introduction

Everyone knows the crucial importance in both pure and applied analysis of the concept of derivative of a function or a distribution discovered by Laurent Schwartz.

Let V be a locally integrable function defined on an open set $\Omega \subset \mathbb{R}^n$ and $v \in \mathbb{R}^n$. We form the differential quotients $\nabla_h V(\cdot,v) = \frac{V(\cdot+hv) - V(\cdot)}{h}$. Instead of requiring that $\nabla_h V(\cdot,v)$ converges for the topology of the pointwise convergence, one is still content with the much weaker convergence of $\nabla_h V(\cdot,v)$, in the space of distributions. In other words, one can find a weak enough topology that allows the convergence of the differential quotients $\nabla_h V(\cdot,v)$.

However, many problems arising in nonlinear analysis, in optimization and differential equations still require the pointwise convergence of the differential quotients $\nabla_h V(\cdot,v)$, but allow to use limsup or liminf instead of the limit. This was already proposed by Dini when V is a locally Lipschitzean function. Few years ago, Clarke suggested to use $\limsup\limits_{\substack{h\to 0+\\ y\to x}} \nabla_h V(y,v)$, whose charm lies in the fact that it is always convex and continuous with respect to v.

We propose in this paper to take in consideration another candidate, namely $\liminf\limits_{\substack{h\to 0+\\ w\to v}} \nabla_h(x,w)$. The main justification for this is that it works well for solving the problems we were studying: we hope to convince the reader by presenting some results in the following pages.

Also, this concept can be defined not only for real valued functions, but can be adapted for vector-valued as well as set-valued maps.

Indeed, one way to see this is to consider the graph of a map. If we can define a tangent space to this graph, then we know that it is the graph of its derivative. If not, we can still define a "tangent cone" to this graph and decide to look at it as the graph of some set-valued map that, hopefully, retains enough properties of a

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would-be derivative to deserve to be presented to the public.

As a seducing candidate, we can think to the Clarke tangent cone, which is always closed and convex. But there is an older candidate, the contingent cone, introduced by Bouligand in the early 1930's, in connection with the theory of derivatives of functions of one or two variables. We claim that it would be unwise to bury and forget it. The contingent cone $D_K(x)$ is defined by

$$(1) \qquad D_K(x) = \bigcap_{\substack{\alpha>0 \\ \varepsilon>0}} \bigcup_{h\in]0,\alpha[} \left(\frac{1}{h}(K-x) + \varepsilon B\right).$$

For this literature, see S. Saks [1], R. T. Rockafellar [5].

We note that when $\operatorname{Int} K \neq \emptyset$ and when $x \in \operatorname{Int} K$, then $D_K(x) = X$. So, conditions involving the contingent cones are boundary conditions, in the sense that they are trivial when $x \in \operatorname{Int} K$. In 1943, Nagumo [1] proved that if a continuous and bounded map f from K to $\mathbb{R}^n$ satisfy

$$(2) \qquad \forall x \in K, \qquad f(x) \in D_K(x)$$

then there exists a trajectory $x(\cdot)$ of the differential equation

$$(3) \qquad x' = f(x), \quad x(0) = x_0 \text{ where } x_0 \text{ is given in } K$$

that remains in the closure of K. Moreover, if for all $x_0 \in K$, there exists a trajectory of the differential equation that remains in K, condition (2) is satisfied.

Analogous statements remain true for differential inclusions (see Haddad [1], Aubin-Cellina-Nohel [1] when K is convex and Aubin-Clarke [1]).

Also, we can use this contingent cone to solve nonlinear equations.

For instance, let f be a continuously differential map from a neighborhood of a compact subset $K \subset \mathbb{R}^n$ to $\mathbb{R}^p$. Assume that

$$(4) \qquad \forall x \in K, \ \exists u \in D_K(x) \text{ such that } \nabla f(x)u = -f(x).$$

Then there exists a solution $\bar{x} \in K$ to the equation $f(x) = 0$ (See Aubin-Clarke [2]). We shall extend this result to inclusions $0 \in F(x)$ as well as finding other results in this direction.

Finally, in optimization theory, contingent cones play a role. For instance, if $x_0 \in K$ minimizes a continuously differentiable function U defined on a neighborhood of K, then

$$(5) \qquad \forall\, u \in D_K(x_0)\ , \qquad \langle \nabla U(x_0), u \rangle \geq 0\ .$$

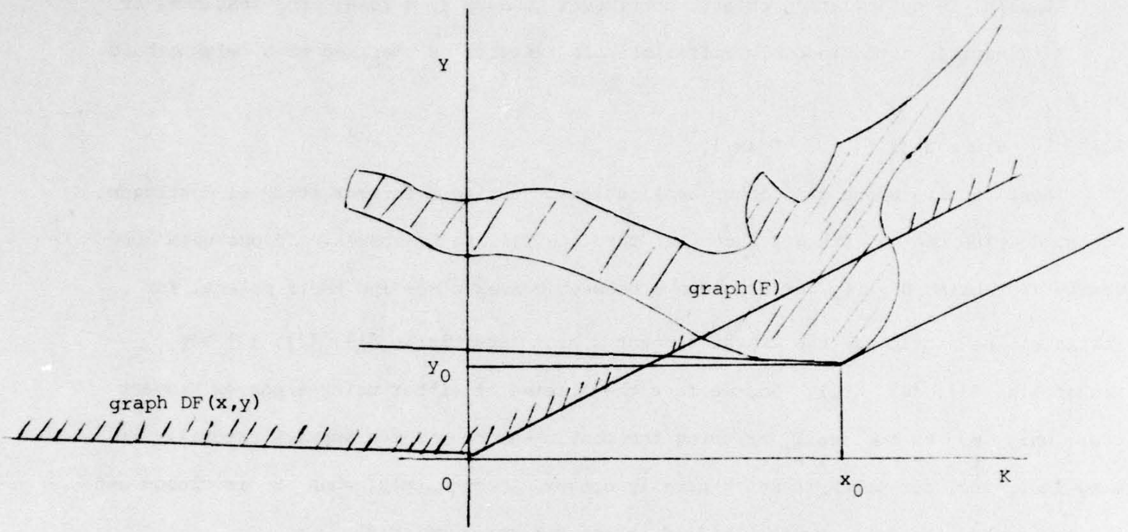
These results among many other applications, justify a further study of contingent cones, despite the unfortunate fact that they can fail to be convex. If one need convexity (for using duality correspondence between convex cones and their polars, for instance), he should use the Clarke tangent cone. (See Clarke [1], [2], [3] and Rockafellar [3], [4], [5]). So, we face the dilemma of either using a convex tangent cone, which may be too small, or using the contingent cone which appears naturally in many instances, but which is not generally convex. Fortunately, when K is closed and convex or when K is a smooth manifold, these two cones coincide.

We proceed as in elementary calculus, when the derivatives of real valued function are defined from the tangents to the graph. Actually, if F is a set-valued map and if (x_0,y_0) belongs to the graph $\mathrm{Graph}(F)$ of F, we can define its contingent cone $D_{\mathrm{graph}(F)}(x_0,y_0)$, which is a closed cone (not necessarily convex). We define the <u>contingent derivative</u> $DF(x_0,y_0)$ of F at x_0 and $y_0 \in F(x_0)$ to be the set-valued map whose graph is $D_{\mathrm{graph}(F)}(x_0,y_0)$. We shall characterize the contingent derivative: $v_0 \in DF(x_0,y_0)(u_0)$ if and only if

$$(6) \qquad \liminf_{\substack{h \to 0+ \\ u \to u_0}} d\left(v_0, \frac{F(x_0 + hu) - y_0}{h}\right) = 0\ .$$

This formula captures the idea of a derivative as a suitable limit of differential quotients.

If one desires to use a concept of derivatives, which would be a set-valued map whose graph is closed and convex (these are called convex processes by Rockafellar [2]), he may use the "Clarke derivative", whose graph is the Clarke tangent cone to the graph of F. (See Ioffe [1] for a similar approach).



Again, the advantages of convexity should be weighted against valuable properties of contingent derivatives in the field of nonlinear equations and differential equations.

What about real-valued functions? Since a real-valued function V is a particular case of a set-valued map, we can define its contingent derivative: $v_0 \in DV(x_0)(u_0)$ if and only if

$$\text{(7)} \qquad \liminf_{\substack{h \to 0+ \\ u \to u_0}} \left| v_0 - \frac{V(x_0+hu) - V(x_0)}{h} \right| = 0 .$$

In many instances, the order relation of real numbers play an important role: this is the case in optimization theory, in differential inequalities and Liapunov stability of trajectories of differential equations or inclusions. In this point of view, it is natural to associate with a real-valued function $x \mapsto V(x)$ the set-valued map $V_+(x) \doteq V(x) + \mathbb{R}_+$ (whose graph is the epigraph of V). So, we check that the contingent derivative $DV_+(x_0, v(x_0))(u_0)$ is the half line $[D_+V(x_0)(u_0), \infty[$ where

$$\text{(8)} \qquad D_+V(x_0)(u_0) = \liminf_{\substack{h \to 0+ \\ u \to u_0}} \frac{V(x_0 + hu) - V(x_0)}{h} .$$

We shall say that $D_+V(x_0)$ is the <u>upper contingent derivative</u> of V. By using the Clarke derivative of V_+ at $(x,V(x))$, we obtain the Clarke generalized directional derivative (see Clarke [1], [2] and Rockafellar [3], [4], [5]). Let us mention that the variational principle holds true. When V is a function from K to $\mathbb{R}$ and when $x_0 \in K$ minimizes V on K, then,

$$\forall\, u \in X\,, \quad 0 \le D_+V(x_0)(u) \tag{9}$$

What makes this property useful is the calculus of upper contingent derivatives for computing $D_+V(x)$ in terms of derivatives of other functions from which V is constructed. As an example, consider the case where $V = U|_K$ is the restriction to K of a continuously differentiable function U. One can prove that

$$D_+V(x_0)(u_0) = \begin{cases} \langle \nabla U(x_0), u_0 \rangle & \text{when } u_0 \in D_K(x_0) \\ +\infty & \text{when } u_0 \notin D_K(x_0) \end{cases} . \tag{10}$$

So, property (9) becomes

$$\forall\, u \in D_K(x_0)\,, \qquad 0 \le \langle \nabla U(x_0), u \rangle \tag{11}$$

or, equivalently, if $D_K(x_0)^-$ denotes the negative polar cone,

$$-\nabla U(x_0) \in D_K(x_0)^- \,. \tag{12}$$

Contingent derivatives do play an important role in sensibility analysis for optimization problems, which is of upmost relevance in economics, for instance. Let $F: K \subset \mathbb{R}^n \to \mathbb{R}^m$ be a compact-valued map and $U: F(K) \times K \to \mathbb{R}$ be a real valued function $(x,y) \to U(x,y)$ which is lower semicontinuous with respect to x. We define the marginal function V by

$$V(y) \doteq \min\{U(x,y) \mid x \in F(y)\} \tag{13}$$

and the marginal (set-valued) map G by

$$G(y) = \{x \in F(y) \mid U(x,y) = V(y)\}\,. \tag{14}$$

We shall prove the following facts:

$\forall\, v_0 \in \operatorname{Dom} DF(y_0,x_0)$, $\forall\, u_0 \in DF(y_0,x_0)(v_0)$ we have

$$D_+ V(y_0)(v_0) + D_+(-U)(x_0,y_0)(u_0,v_0) \le 0 \tag{15}$$

and, for the contingent derivative of the marginal map G, $\forall\, v_0 \in \operatorname{Dom} DG(y_0,x_0)$, $\forall\, u_0 \in DG(y_0,x_0)(v_0)$, we have

(16) $$D_+ U(x_0,y_0)(u_0,v_0) + D_+(-V)(y_0)(v_0) \leq 0.$$

In this line of thought, we can state a differential version of Ekeland's variational principle: Let K be a closed subset and $V:K \to [0,\infty[$ a lower semicontinuous function. Then we can associate with any $\varepsilon > 0$ and $x_\varepsilon \in K$ satisfying $V(x_\varepsilon) \leq \inf V(x) + \varepsilon^2$ an element $\bar{x}_\varepsilon \in K$ which satisfies

(17) $$\begin{cases} \text{i)} & \|x_\varepsilon - \bar{x}_\varepsilon\| \leq \varepsilon \\ \text{ii)} & \forall\, u \in X\ ,\quad 0 \leq D_+V(\bar{x}_\varepsilon)(u) + \varepsilon\|u\|\ . \end{cases}$$

This result is as useful as the original version of Ekeland's theorem. It yields surjectivity theorems and inverse function theorems. For instance, we shall prove that when an upper semicontinuous map F from a closed subset $K \subset X$ to the compact subsets of Y satisfies

(18) $$\begin{cases} \exists\, c > 0 \text{ such that, } \forall\, x \in K\ ,\ \forall\, y \in F(x),\ \forall\, v \in Y\ , \\ \exists\, u \in X \text{ satisfying } v \in DF(x,y)(u) \text{ and } c\|u\| \leq \|v\| \end{cases}$$

then F maps K <u>onto</u> Y.

We also shall use this kind of approach for solving inclusions $0 \in F(x_*)$. (We shall say that x_* is a stationary point of F.)

We introduce two functions

(19) $$V:K \to \mathbb{R}_+ \quad\text{and}\quad W:K \times \overline{co}(F(x)) \to \mathbb{R}_+$$

and we shall say that V is a Liapunov function for F with respect to W if

(20) $$\forall\, x \in K,\ \exists\, v \in F(x) \text{ such that } D_+V(x)(v) + W(x,v) \leq 0\ .$$

We shall observe that when V is lower semicontinuous and lower semicompact (i.e., the subsets $\{x \in K \mid V(x) \leq \lambda\}$ are relatively compact for all $\lambda \in \mathbb{R}$),

(21) there exists $x_* \in K$ and $v_* \in F(x_*)$ such that $W(x_*,v_*) = 0$.

We note that when we assume that $W(x_*,v_*) = 0$ if and only if $v_* = 0$, such an x_* is a stationary point of F. This is not all. Assume, for instance, that

(22) $$\begin{cases} \text{i)} & F \text{ is bounded, upper semicontinuous and has compact convex values} \\ \text{ii)} & V \text{ is continuous and lower semicompact} \\ \text{iii)} & W \text{ is continuous and is convex with respect to } v\ . \end{cases}$$

We shall prove that V is a Liapunov function for F with resepct to W if and only if for all $x_0 \in K$, there exists a trajectory $x(\cdot) \in \mathcal{C}(0,\infty;\mathbb{R}^n)$ of the differential inclusion

$$(23) \qquad x' \in F(x) \ , \quad x(0) = x_0$$

which is monotone in the sense that

$$(24) \qquad \forall\, s > t, \quad V(x(s)) - V(x(t)) + \int_s^t W(x(\tau),x'(\tau))d\tau \le 0 \ .$$

In this case, subsequences $x(t_n)$ and $x'(t_n)$ have "almost cluster points" $x_* \in K$ and $v_* \in F(x_*)$ satisfying property (22), where "almost cluster points" are analogs for measurable classes of functions of cluster points for usual function. Monotone trajectories yield informations on the behavior of the nonincreasing function $t \to V(x(t))$ when $t \to \infty$. We also prove that under assumptions (22) i) and iii), the function V_F defined on K by

$$(25) \qquad V_F(x_0) = \inf \{\int_0^\infty W(x(\tau),x'(\tau))d\tau \quad \text{when} \quad x(\cdot) \text{ is a solution to (23) }\},$$

is the smallest Liapunov function for F with respect to W when Liapunov functions do exist. This provides a bridge between Liapunov stability theory and Caratheodory-Bellman approach to optimal control theory.

I would like to thank Georges Haddad for his hidden but important contribution as well as Arrigo Cellina, Bernard Cornet and Ivar Ekeland.

OUTLINE

1. Bouligand's contingent cones.
2. Calculus on contingent cones.
3. Contingent derivative of a set-valued map.
4. Calculus on contingent derivatives.
5. Upper contingent derivative of a real-valued function.
6. Calculus on upper contingent derivatives.
7. Contingent derivatives of marginal functions and marginal maps.
8. Ekeland's variational principle.
9. Surjectivity thecrems.
10. The Newton method.
11. Liapunov functions and existence of stationary points.
12. Monotone trajectories of a differential inclusion.
13. Almost convergence of monotone trajectories to stationary points.
14. Necessary conditions for the existence of monotone trajectories.
15. Sufficient conditions for the existence of monotone trajectories.
16. Stability and asymptotic stability.
17. Liapunov functions for U-monotone maps.
18. Construction of Liapunov functions.
19. Construction of dynamical systems having monotone trajectories.
20. Feedback controls yielding monotone trajectories.
21. The time dependent case.

1. Bouligand's contingent cone.

Let K be a nonempty subset of a Hilbert space X.[1] We shall define the Bouligand contingent cone as follows.

Definition 1

We say that the subset

$$(1) \qquad D_K(x) = \bigcap_{\varepsilon>0} \bigcap_{\alpha>0} \bigcup_{0<h\leq \alpha} \left(\frac{1}{h}(K-x) + \varepsilon B\right)$$

is the "contingent cone" to K at x. ■

In other words, $v \in D_K(x)$ if and only if

$$(2) \qquad \begin{cases} \forall\, \varepsilon > 0\ ,\ \forall\, \alpha > 0\ ,\ \exists\, u \in v + \varepsilon B\ ,\ \exists\, h \in\,]0,\alpha] \text{ such that} \\ x + hu \in K\ . \end{cases}$$

It is quite obvious that $D_K(x)$ is a closed cone, which is contained in the closed cone $T_K(x)$ defined by

$$T_K(x) \doteq c\ell\Big(\bigcup_{h>0} \frac{1}{h}(K-x)\Big)$$

They coincide when K is a closed convex subset. (See, for instance, Rockafellar [5]).

We also note that

$$(3) \qquad \text{if } x \in \operatorname{Int}(K), \text{ then } D_K(x) = X\ .$$

We characterize the contingent cone by using the distance function $d_K(\cdot)$ to K defined by

$$d_K(x) \doteq \inf\{\|x-y\| \mid y \in K\}.$$

Proposition 1

$$v \in D_K(x) \text{ if and only if } \liminf_{h\to 0+} \frac{d_K(x+hv)}{h} = 0\ . \qquad ■$$

Proof.

a). Let $v \in D_K(x)$. For all $\varepsilon > 0$, $\alpha > 0$, there exist $h \in\,]0,\alpha]$ and $u \in v + \varepsilon B$ such that $x + h u \in K$. *Hence* $\frac{d_K(x+hv)}{h} \leq \frac{1}{h}\|x + hv - (x+hu)\| \leq \|u-v\| \leq \varepsilon$. So, $\forall\, \varepsilon > 0$, $0 \leq \sup_{\alpha} \inf_{h\leq\alpha} \frac{d_K(x+hv)}{h} \leq \varepsilon$. This proves that $\liminf_{h\to 0+} \frac{d_K(x+hv)}{h} = 0$.

b). Conversely, if $\liminf_{h\to 0+} \frac{d_K(x+hv)}{h} = \sup_{\alpha>0} \inf_{h<\alpha} \frac{d_K(x+hv)}{h} = 0$, we deduce that

[1] For simplicity. Several results of this paper are true for topological vector spaces.

$\forall\, \varepsilon > 0$, $\forall\, \alpha > 0$, $\exists\, h \leq \alpha$ such that $\dfrac{d_K(x+hv)}{h} \leq \varepsilon/2$. Thus, there exists $y \in K$ such that $\dfrac{\|x+hv-y\|}{h} \leq \dfrac{d_K(x+hv)}{h} + \varepsilon/2$. Hence $u = \dfrac{y-u}{h} \in v + \varepsilon B$ and satisfies $x + hu = y \in K$. ■

Remark.

We recognize the "Nagumo condition" implying the existence of trajectories remaining in a given subset K . ■

We can also characterize the contingent cone in terms of sequences.

Proposition 2

$v \in D_K(x)$ if and only if there exists a sequence of strictly positive numbers h_n and of elements $u_n \in X$ satisfying

(4) i) $\lim\limits_{n\to\infty} u_n = v$, ii) $\lim\limits_{n\to\infty} h_n = 0$, iii) $\forall\, n \geq 0$, $x + h_n u_n \in K$.

Proof. It is left as an exercise.

Remark.

For all $x \in X$, we have $D_X(x) = X$. We shall set $D_\emptyset(x) \doteq \emptyset$. ■

Remark.

It is easy to see that the contingent cone to K and the contingent cone to the closure $\bar{K}$ of K coincide:

$$\forall\, x \in K\ ,\quad D_K(x) = D_{\bar{K}}(x)\ .$$

Therefore, there is no danger in speaking of $D_K(x)$ even when $x \in \bar{K}$ and $x \notin K$. ■

Proposition 3

Let $K \subset X$ be a closed subset. We denote by $\pi_K(y)$ the subset of elements $x \in K$ such that $\|x-y\| = d_K(y)$. We obtain the following inequalities

(5) $\forall\, y \notin K$, $\forall\, x \in \pi_K(y)$, $\forall\, v \in \overline{co}\, D_K(x)$, then $\langle y - x, v \rangle \leq 0$. ■

Proof.

Let $x \in \pi_K(y)$ and $v \in D_K(x)$. We deduce from the inequalities $\|y-x\| - d_K(x + hv) = d_K(y) - d_K(x + hv) \leq \|y - x - hv\|$ that

$$\frac{\langle y - x, v \rangle}{\| y-x \|} = \lim_{h \to 0+} \frac{\|y-x\| - \|y-x-hv\|}{h} \leq \liminf_{h \to 0+} \frac{d_K(x+hv)}{h} = 0$$

for $y \neq x$, $u \to \|u\|$ is differentiable at $u \neq 0$ and $v \in D_K(x)$. So $\langle y - x, v \rangle \leq 0$ for all $v \in D_K(x)$, and, consequently, for all $v \in \overline{co}\, D_K(x)$. ■

We deduce from this proposition a criterion of convexity of the contingent cone. Let us recall that a set-valued map F from M to N is lower semicontinuous at $x_0 \in M$ if for any $\varepsilon > 0$ and for any $y_0 \in F(x_0)$ there exists $\eta > 0$ such that, $F(x) \cap (y_0 + \varepsilon B) \neq \emptyset$ for all $x \in x_0 + \eta B$.

<u>Theorem 1</u> (B. Cornet)

Let us assume that

(6) $x \in \bar{K} \mapsto \overline{co}\, D_K(x)$ is lower semicontinuous at $x_0 \in \bar{K}$.

Then the contingent cone $D_K(x_0)$ to K at x_0 is a closed convex cone. ■

<u>Proof</u>.

a) Let $v_0 \in \overline{co}\, D_K(x_0)$. For proving that $v_0 \in D_K(x_0)$, fix $\varepsilon > 0$ and let $\eta \doteq \eta(\varepsilon)$ such that, thanks to (6),

$$d_{\overline{co}\, D_K(x)}(v_0) \leq d_{\overline{co}\, D_K(x_0)}(v_0) + \varepsilon/2 \quad \text{when} \quad \|x-x_0\| \leq \eta .$$

We take $h \doteq \eta/2\|v_0\|$ and, for $t \in]0,h[$, we set $y_t \doteq x_0 + t\, v_0$, we choose $x_t \in \pi_K(y_t)$ and $v_t \doteq \pi_{\overline{co}D_K(x_t)}(x_0)$. Hence Proposition 3 implies that

(7) $$\langle y_t - x_t, v_t \rangle \leq 0 .$$

We observe that $\|x_t-x_0\| \leq \|x_t-y_t\| + \|y_t-x_0\| \leq 2\|y_t-x_0\|$ (for $x_t \in \pi_K(y_t)$). Consequently,

(8) $$\|x_t-y_t\| \leq 2t\|v_0\| \leq \eta \quad \text{when} \quad t \leq h$$

and thus,

(9) $$\|v_t-v_0\| = d_{\overline{co}\, D_K(x_t)}(v_0) \leq \varepsilon/2.$$

Hence, inequalities (7), (8) and (9) imply

(10) $$\langle y_t-x_t, v_0 \rangle \leq \langle y_t-x_t, v_0-v_t \rangle + \langle y_t-x_t, v_t \rangle \leq \|y_t-x_t\|\, \|v_0-v_t\| \leq 2t\varepsilon/2 .$$

Let us set

$$f(t) \doteq \tfrac{1}{2} d_K(x_0+tv)^2 .$$

It is a locally Lipschitzean function, which is thus almost everywhere differentiable. If $t \in]0,h[$ is such that $f'(t)$ exists, we obtain

$$\begin{aligned} f'(t) &= \lim_{\theta \to 0+} \frac{1}{2\theta}(d_K(y_t + \theta v_0)^2 - d_K(y_t)^2) \\ &\leq \lim_{0+} \frac{1}{2\theta}(\|y_t-x_t + \theta v_0\|^2 - \|y_t-x_t\|^2) \quad \text{for } x_t \in \pi_K(y_t) \\ &= \langle y_t-x_t, v_0 \rangle \leq t\,\varepsilon \quad \text{(by (10))}. \end{aligned}$$

Hence,

$$\frac{1}{2} d_K(x_0+h\, v_0)^2 = f(h) - f(0) = \int_0^h f'(t)dt \leq \varepsilon \int_0^h t dt = \frac{\varepsilon h^2}{2} .$$

This implies that

$$\liminf_{h\to 0+} \frac{d_K(x_0+hv_0)}{h} \leq \sqrt{\varepsilon} .$$

Since ε was chosen arbitrarily, we deduce that $\liminf_{h\to 0+} \frac{d_K(x_0+hv_0)}{h} = 0$, i.e., that $v_0 \in D_K(x_0)$. We have proved that $\overline{co}\, D_K(x_0) = D_K(x_0)$. ∎

We mention the following consequence.

Theorem 2 (Cornet)

Let us assume that X is finite dimensional and that

(11) $\qquad x \in \bar{K} \mapsto D_K(x)$ is lower semicontinuous at x_0 .

Then $D_K(x_0)$ is a closed convex cone and

(12) $$\forall\, v_0 \in D_K(x_0) , \quad \lim_{\substack{x\to x_0 \\ x\in K}} \liminf_{h\to 0+} \frac{d_K(x+hv_0)}{h} = 0$$ ∎

Proof.

We recall that the lower semicontinuity of $x \mapsto D_K(x)$ at x_0 implies that the negative polar cone $D_K(\cdot)^-$ satisfies:

For all sequence of elements $x_n \in K$ converging to $x_0 \in K$ and for all sequence of elements $p_n \in N_K(x_0)$ converging to p_0 , we have $p_0 \in N_K(x_0)$.

When X is finite dimensional, this latter property implies that the set-valued map $x \mapsto \overline{co}\, D_K(x)$ is lower semicontinuous at $x_0 \in K$. (See J. P. Aubin and A. Cellina [1]).

Hence $D_K(x_0)$ is a closed convex cone by Theorem 1. Let v_0 be chosen in $D_K(x_0)$ and let us associate with any $x \in \bar{K}$ an element $v \in D_K(x)$ such that $\|v_0-v\| = d_{D_K(x)}(v_0)$. Since $x \mapsto D_K(x)$ is lower semicontinuous at x_0, there exists $\eta > 0$ such that $\|v-v_0\| = d_{D_K(x)}(v_0) \leq d_{D_K(x_0)}(v_0) + \varepsilon = \varepsilon$ whenever $\|x-x_0\| \leq \eta$. Therefore for all $h > 0$,

$$\frac{d_K(x+hv_0) - d_K(x+hv)}{h} \leq \|v-v_0\| \leq \varepsilon.$$

Since $v \in D_K(x)$, we deduce that for all $x \in x_0 + \eta B$,

$$\liminf_{h\to 0+} \frac{d_K(x+hv_0)}{h} \leq \liminf_{h\to 0+} \frac{d_K(x+hv)}{h} + \varepsilon = \varepsilon.$$ ∎

2. Calculus on contingent cones.

We state and prove several properties of contingent cones.

Proposition 1

Let $K \subset L \subset X$ be two nonempty subsets. Then,

$$(1) \qquad \forall x \in K, \quad D_K(x) \subset D_L(x) . \qquad \blacksquare$$

Proof. It is left as an exercise.

Proposition 2 (Hess)

Let $K = \bigcup_{i \in I} K_i$ be the union of subsets K_i. If $x \in K$, we set $I(x) = \{i \in I$ such that $x \in K_i\}$. Then

$$(2) \qquad \forall x \in K, \quad \bigcup_{i \in I(x)} D_{K_i}(x) \subset D_K(x) .$$

If I is finite or, more generally, locally finite in the sense that there exists $r > 0$ such that $x + rB$ meets only a finite number of K_i, and if the subsets K_i are closed, then

$$(3) \qquad \bigcup_{i \in I(x)} D_{K_i}(x) = D_K(x) .$$

Proof.

The first inclusion is obvious. For proving the opposite inclusion, take $v \in D_K(x)$. Then, there exists a sequence of elements $u_n \in X$ and $h_n > 0$ such that $x + h_n u_n \in K$, $\lim_{h\to\infty} h_n = 0$ and $\lim_{h\to\infty} u_n = v$. There exists n_r such that, for all $n \geq n_r$, $x + h_n u_n \in x + rB$. Let $I_r(x) \doteq \{i \in I \mid (x + rB) \cap K_i \neq \emptyset\}$, which is finite by assumption. Then, $\forall n \geq n_r$, $x + h_n u_n \in \bigcup_{i \in I_r(x)} K_i$. Therefore, there exists at least an index $i_0 \in I_r(x)$ and an infinite subsequence such that

$$x + h_{n_p} u_{n_p} \in K_{i_0} .$$

This proves two facts. First, that $v \in D_{K_{i_0}}(x)$ and second, that $x \in c\ell(K_{i_0}) = K_{i_0}$ (by assumption). Hence $i_0 \in I(x)$.

Proposition 3

Let $K \doteq \bigcap_{i \in I} K_i$ be the intersection of subsets K_i. We set $J(x) \doteq \{i \in I$ such that $x \notin \mathrm{Int}\, K_i\}$. Then

$$(4) \qquad \forall x \in K, \quad D_K(x) \subset \bigcap_{i \in J(x)} D_{K_i}(x) . \qquad \blacksquare$$

Proof. It is left as an exercise.

Proposition 4

Let $K = \prod_{i \in I} K_i$ be the product of a family of nonempty subsets K_i of Hilbert spaces X_i . Then

(5) $$\forall x \in K\ ,\quad D_K(x) \subset \prod_{i \in I} D_{K_i}(x_i)\ . \qquad \blacksquare$$

Proof. It is left as an exercise.

The following proposition gives some information about the contingent cone of the image of a subset by a smooth map.

Proposition 5

Let X and Y be two Hilbert spaces, $K \subset X$ be a subset of X and A be a continuously differentiable map from an open neighborhood of K to Y . Then

(6) $$\forall x \in K\ ,\quad \nabla A(x) D_K(x) \subset D_{A(K)}(Ax)\ . \qquad \blacksquare$$

Proof. Let $v \in D_K(x)$, $\varepsilon > 0$ and $\alpha > 0$. Since A is continuously differentiable, there exist $\beta > 0$, $\eta > 0$ such that, $\forall\, h \leq \beta$ and for all $u \in v + \eta B$, $A(x + hu) = A(x) + h\nabla A(x)u + he$ where $e \in \frac{\varepsilon}{2} B$. Let $\delta = \min(\frac{\varepsilon}{2\|\nabla A(x)\|}, \eta)$ and $\gamma = \min(\alpha, \beta)$. Since $v \in D_K(x)$, there exists $h < \gamma$ and $u \in v + \delta B$ such that $x + hu \in K$. Hence $A(x) + h(\nabla A(x)u + e) = A(x + hu) \in A(K)$ and $\|\nabla A(x)u + e - \nabla A(x)v\| \leq \varepsilon/2 + \|\nabla A(x)\|\delta \leq \varepsilon$. This states that $\nabla A(x)v \in D_{A(K)}(Ax)$. $\blacksquare$

In particular, if $A \in \mathcal{L}(X,Y)$, we obtain the formula

(7) $$\forall x \in K\ ,\quad A\, D_K(x) \subset D_{A(K)}(A(x))$$

We study now the contingent cone to the preimage of a set by a smooth map:

Proposition 6

Let X and Y be two Hilbert spaces, $L \subset X$ and $M \subset Y$ be two subsets and A be a continuously differentiable map from an open neighborhood of L to Y . We set

(8) $$K \doteq \{x \in L \mid A(x) \in M\} = L \cap A^{-1}(M)\ .$$

Then,

(9) $$\forall x \in K,\ D_K(x) \subset D_L(x) \cap \nabla A(x)^{-1} \cdot D_M(A(x))\ . \qquad \blacksquare$$

Proof.

By Proposition 1, $D_K(x) \subset D_L(x)$ because $K \subset L$. By Proposition 5, $\nabla A(x) D_K(x) \subset D_{A(K)}(Ax) \subset D_M(Ax)$ because $A(K) \subset M$. Hence $D_K(x) \subset \nabla A(x)^{-1} D_M(Ax)$ and consequently, formula (9) holds true.

3. Contingent derivative of a set-valued map.

We adapt to the case of a set-valued map the intuitive definition of a derivative of a function in terms of the tangent to its graph.

Let F be a proper set-valued map from $K \subset X$ to Y. (We say that F is proper if its images $F(x)$ are nonempty for all $x \in K$). Let $x_0 \in K$ and $y_0 \in F(x_0)$.

We denote by $DF(x_0,y_0)$ the set-valued map from X to Y whose graph is the contingent cone $D_{\text{graph}(F)}(x_0,y_0)$ to the graph of F at (x_0,y_0). ∎

In other words,

(1) $$v_0 \in DF(x_0,y_0)(u_0) \text{ if and only if } (u_0,v_0) \in D_{\text{graph}(F)}(x_0,y_0).$$

Definition 1

We shall say that the set-valued map $DF(x_0,y_0)$ from X to Y is the "contingent derivative" of F at $x_0 \in K$ and $y_0 \in F(x_0)$. ∎

It is a positively homogeneous set-valued map (since its graph is a cone) with closed graph. Also, we note that

(2) $$\text{Dom } DF(x_0,y_0) \subset D_K(x_0)$$

i.e., that the domain of $DF(x_0,y_0)$ is contained in the contingent cone to K at x_0.

We first point out that

(3) $$\forall x_0 \in K, \ \forall y_0 \in F(x_0), \ DF(x_0,y_0)^{-1} = D(F^{-1})(y_0,x_0).$$

Indeed, to say that $(u_0,v_0) \in D_{\text{graph}(F)}(x_0,y_0)$ amounts to saying that $(v_0,u_0) \in D_{\text{graph}(F^{-1})}(y_0,x_0)$. ∎

Example: Indicator of a set and its contingent derivative

Among the set-valued maps from X to a Hilbert space Y, we single out the "indicator" of K which is the set-valued map Ψ_K from X to Y defined by

$$\Psi_K(x) = \begin{cases} 0 & \text{if } x \in K \\ \emptyset & \text{if } x \notin K. \end{cases}$$

Note that Ψ_K depends upon the choice of the Hilbert space Y. We recall the following conventions

If $M \subset Y$, then $M + \emptyset = \emptyset + M = \emptyset$

and

$d(x,\emptyset) = +\infty$.

Note also that if F is a set-valued map from X to Y , then $F + \Psi_K$ yields the restriction of F to K (since $(F+\Psi_K)(x) = \emptyset$ when $x \notin K$).

<u>Proposition</u> 1

The contingent derivative of the indicator of a set K is the indicator of the contingent cone to K :

(4) $$\forall\, x \in K\ ,\quad D\Psi_K(x) = \Psi_{D_K(x)}\ . \qquad \blacksquare$$

<u>Proof</u>.

$$\text{Graph}\,[D\psi_K(x)] = D_{\text{Graph}(\psi_K)}(x,0) = D_{K\times\{0\}}(x,0)\ .$$

It is easy to check that $D_{K\times\{0\}}(x,0) = D_K(x) \times \{0\}$. So, $\text{Graph}\,[D\psi_K(x)] = D_K(x) \times \{0\} :=$ $\text{Graph}\,[\psi D_K(x)]$. $\blacksquare$

<u>Characterization of the contingent derivative</u>

We shall give an analytical characterization of $DF(x_0,y_0)$, which justifies that the above definition is a reasonable candidate for capturing the idea of a derivative as a (suitable) limit of differential quotients. We extend F to X by setting $F(x) = \emptyset$ when $x \notin K$.

<u>Theorem</u> 1

Let F be a set-valued map from $K \subset X$ <u>to</u> Y and $(x_0,y_0) \in \text{graph}(F)$. Then $v_0 \in DF(x_0,y_0)(u_0)$ if and only if

(5) $$\liminf_{\substack{h \to 0+ \\ u \to u_0}} d\left(v_0, \frac{F(x_0+hu) - y_0}{h}\right) = 0 \qquad \blacksquare$$

<u>Proof</u>.

To say that $v_0 \in DF(x_0,y_0)(u_0)$, i.e., that $(u_0,v_0) \in D_{\text{graph}(F)}(x_0,y_0)$, amounts to saying that for all $\varepsilon_1,\varepsilon_2 > 0$ and $\alpha > 0$, there exists $u \in u_0 + \varepsilon_1 B$ and $v \in v_0 + \varepsilon_2 B$ such that $v \in \dfrac{F(x_0 + hu) - y_0}{h}$.

This is equivalent to say that $\forall\ \varepsilon_1 > 0\ ,\ \varepsilon_2 > 0\ ,\ \alpha > 0$, we have

$$\inf_{h \leq \alpha}\ \inf_{\|u-u_0\| \leq \varepsilon_1} d\left(v_0, \frac{F(x_0+hu) - y_0}{h}\right) \leq \varepsilon_2$$

The last statement is equivalent to (5). ∎

When F is a single valued map, we set

(6) $$DF(x_0,y_0) = DF(x_0)$$

since $y_0 = F(x_0)$. The above formula shows that in this case, $v_0 \in DF(x_0)(u_0)$ if and only if

(7) $$\liminf_{\substack{h \to 0+ \\ u \to u_0}} \frac{\|F(x_0+hu) - F(x_0) - hv_0\|}{h} = 0.$$

If F is a single-valued map which is regularly Gâteaux differentiable, in the sense that there exists $\nabla F(x) \in \mathcal{L}(X,Y)$ satisfying:

(8) $$\forall\ u_0 \in X,\ \lim_{\substack{h \to 0+ \\ u \to u_0}} \frac{F(x_0+hu) - F(x_0)}{h} = \nabla F(x_0)u_0$$

then the contingent derivative coincides with the Gâteaux derivative:

(9) $$\forall\ u_0 \in X\ ,\ DF(x_0)(u_0) = \nabla F(x_0)u_0\ .$$

Example : Contingent derivatives of locally Lipschitzean maps.

This formula has a simpler form when the set-valued map F is upper locally Lipschitzean:

Definition 2

We say that a set-valued map F is "upper locally Lipschitzean at $x_0 \in \operatorname{Int} K$" if there exists a neighborhood $N(x_0)$ of x_0 and a constant $\ell > 0$ such that

(10) $$\forall\ x \in N(x_0)\ ,\ F(x) \subset F(x_0) + \ell\|x-x_0\|B\ .$$ ∎

Naturally, any locally Lipschitzean (and, a fortiori, Litschitzean) set-valued map is upper locally Lipschitzean.

Proposition 2

Let F be an upper locally Lipschitzean set-valued map from $\operatorname{Int} K \subset X$ to Y , $x_0 \in \operatorname{Int} K$ and $y_0 \in F(x_0)$. Then $v_0 \in DF(x_0,y_0)(u_0)$ if and only if

(11) $$\liminf_{h \to 0+} d\left(v_0, \frac{F(x_0+h\, u_0) - y_0}{h}\right) = 0$$

Remark.

If F is a locally Lipschitzean single-valued map, this formula becomes

(12) $$\liminf_{h \to 0+} \frac{\|F(x_0+h\, u_0) - F(x_0) - h.v_0\|}{h} = 0$$ ∎

Proof.

Let $v_0 \in DF(x_0,y_0)(u_0)$. Then for all $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\alpha > 0$, there exists $h < \alpha$ such that

$$v_0 \in \bigcup_{\|u-u_0\| \leq \varepsilon_1} \frac{F(x_0+hu)-y_0}{h} + \varepsilon_2 B .$$

But, F being upper locally Lipschitzean, we know that

$$\begin{aligned} F(x_0 + hu) &\subset F(x_0 + h\, u_0) + \ell h\|u-u_0\| \\ &\subset F(x_0 + h\, u_0) + \ell h \varepsilon_1 . \end{aligned}$$

Hence, for all $\varepsilon_1, \varepsilon_2 > 0$,

$$v_0 \in \bigcap_{\alpha>0} \bigcup_{h \leq \alpha} \left(\frac{F(x_0+h\, u_0) - y_0}{h} + (\varepsilon_1 \ell + \varepsilon_2)B \right) .$$

This implies formulas (11). ∎

4. Calculus on contingent derivatives.

We shall derive from the properties of the contingent cones a calculus on contingent derivatives of set-valued maps. We shall start naturally with the chain rule:

Proposition 1

Let F be a set-valued map from $K \subset X$ to Y and A be a continuously differentiable map from an open neighborhood of $F(K) \subset Y$ to Z. Then

(1) $\forall u_0 \in X, \quad \nabla A(y_0) \cdot DF(x_0,y_0)(u_0) \subset D(AF)(x_0,Ay_0)(u_0)$ ■

Proof.

Let $(1 \times A)$ be the map: $(x,y) \in X \times F(K) \mapsto (x,A(y)) \in X \times Z$. The graph G of the set-valued map $x \mapsto AF(x)$ is related to the graph F of F by the relation $G = (1 \times A)F$. By Proposition 2.5, we know that $(1 \times \nabla A(y_0))D_F(x_0,y_0) \subset D_G(x_0,Ay_0)$. This states that for all $v_0 \in DF(x_0,y_0)(u_0)$, $\nabla A(y_0)v_0 \in D(FA)(x_0,Ay_0)(u_0)$. ■

Proposition 2

Let F be a set-valued map from $K \subset X$ to Y and A be a continuously differentiable map from an open subset $\Omega \subset Z$ from X. *Then,*

(2) $\forall u_0 \in Z, \quad D(FA)(x_0,y_0)(u_0) \subset DF(Ax_0,y_0)(\nabla A(x_0)(u_0))$. ■

Proof.

Let $(A \times 1):\Omega \times Y \to K \times Y$ be the map defined by $(A \times 1)(z,y) = (Az,y)$. The graph G of the set-valued map FA from Ω to Y is related to the graph F of F by the relation $G = (A \times 1)^{-1}F$. By Proposition 2.6, we know that $D_G(x_0,y_0) \subset (\nabla A(x_0) \times 1)^{-1}$ $D_F(Ax_0,y_0)$. This states that for all $u_0 \in Z$ and $v_0 \in D(FA)(x_0,y_0)(u_0)$, then $v_0 \in DF(Ax_0,y_0))(\nabla A(x_0)(u_0))$. ■

We state now the properties of contingent derivatives of unions and intersections of maps.

Proposition 3

Let us consider a family of set-valued maps $F_i (i \in I)$ from K to X and set $F(x) = \bigcup_{i \in I} F_i(x)$. We associate with any $x_0 \in K$ and $y_0 \in F(x_0)$ the subset $I(x_0,y_0) = \{i \in I \mid y_0 \in F_i(x_0)\}$. Then

(3) $$\forall u_0 \in X\ ,\quad \bigcup_{i \in I(x_0,y_0)} DF_i(x_0,y_0)(u_0) \subset DF(x_0,y_0)(u_0)\ .$$

Equality holds when the graphs of the maps F_i are closed and when the family is locally finite. ∎

Proof.

We note that the graph $\mathcal{F}$ of F is the union of the graphs $\mathcal{F}_i$ of F_i. Hence Proposition 3 follows from Proposition 2.2.

Proposition 4

Let us consider a family of set-valued maps F_i $(i \in I)$ from K to X such that $F(x) \doteq \bigcap_{i \in I} F_i(x) \neq \emptyset$ for all $x \in K$. Let $J(x_0,y_0) = \{i \in I$ such that $(x_0,y_0) \notin \mathrm{int}(\mathrm{graph}(F_i))\}$. Then

(4) $$\forall u_0 \in X\ ,\quad DF(x_0,y_0)(u_0) \subset \bigcap_{i \in J(x_0,y_0)} DF_i(x_0,y_0)(u_0)\ .$$ ∎

Proof

We note that the graph of F is the intersection of the graphs of the maps F_i and we apply Proposition 2.3.

We turn our attention to the study of contingent derivatives of restrictions. First, we begin with this remark.

Proposition 5

If $F \subset G$, in the sense that graph $F \subset$ graph G, then $\forall (x_0,y_0) \in \mathrm{graph}(F)$, we have

(5) $$DF(x_0,y_0)(u_0) \subset DG(x_0,y_0)(u_0)$$ ∎

Proof.

It follows from the fact that $D_{\mathrm{graph}(F)}(x_0,y_0) \subset D_{\mathrm{graph}(G)}(x_0,y_0)$. ∎

In particular, if $f(\cdot)$ is a regularly Gâteaux-differentiable selection of $F(\cdot)$, its Gâteaux-derivative $\nabla f(x)$ is a selection of the set-valued map $DF(x,f(x))$. ∎

We note that the restriction $F|_L \doteq F + \Psi_L$ of a set-valued map F to a subset L is contained in F. In this case, we obtain a more precise result.

Proposition 6

Let F be a set-valued map from $K \subset X$ to Y, L be a subset of K and $F|_L$ be the restriction of F to L. Then, for any $x_0 \in L$ and $y_0 \in F(x_0)$,

$$DF\big|_L(x_0,y_0)(u_0) \subset DF(x_0,y_0)\big|_{D_L(x_0)}(u_0). \tag{6}$$ ∎

In other words, the contingent derivative of the restriction of F to L is contained in the restriction of the contingent derivative to the contingent cone to L.

Proof.

Indeed, $v_0 \in DF|_L(x_0,y_0)(u_0)$ if and only if for all $\varepsilon > 0$, for all $\alpha > 0$, there exists $h > 0$ such that

$$v_0 \in \bigcup_{\substack{\|u-u_0\| \leq \varepsilon \\ u \in L}} \frac{F|_L(x_0+h\,u) - y_0}{h} \subset \bigcup_{\|u-u_0\| \leq \varepsilon} \frac{F(x_0+h\,u) - y_0}{h}$$

This requires that there exists $u \in u_0 + \varepsilon B$ such that $x + hu \in L$. Hence u_0 belongs necessarily to $D_L(u_0)$ and formula (6) ensues. ∎

We give now a formula on the contingent derivative of the sum of a set-valued map and a smooth single-valued map.

Proposition 7

Let us assume that the set-valued map F is defined on K by $F(x) = G(x) + H(x)$ where H is a continuously differentiable map from a neighborhood of K to Y and G is any set-valued map from K to Y. Then, for any $x_0 \in K$, $z_0 \in G(x_0)$ and $y_0 = z_0 + H(x_0) \in F(x_0)$, we have:

$$DG(x_0,z_0)(u_0) + \nabla H(x_0)\cdot u_0 \subset DF(x_0,y_0)(u_0). \tag{7}$$ ∎

Proof.

Let $u_0 \in \mathrm{Dom}\, DG(x_0,y_0)$ be given and $v_0 \in DG(x_0,y_0)(u_0)$. Since H is continuously differentiable, for any $\varepsilon > 0$, there exist $\delta \in]0,\varepsilon]$ and $\alpha > 0$ such that for all $u \in u_0 + \delta B$ and for all $h \leq \alpha$, $H(x_0 + hu) = H(x_0) + h\nabla H(x_0)u + h\,e$, with $\|e\| \leq \varepsilon$. Since $v_0 \in DG(x_0,y_0)(u_0)$, there exists $u \in u_0 + \delta B$ and $h \leq \alpha$ such that $v_0 \in \frac{G(x_0 + hu) - z_0}{h} + \varepsilon B$. Hence $v_0 + \nabla H(x_0)u_0 \in \frac{F(x_0+hu)-z_0}{h} + 2\varepsilon B + \nabla H(x_0)(u_0-u) \in \frac{F(x_0+hu) - z_0}{h} + \varepsilon(2 + \|\nabla H(x_0)\|)B$. ∎

By taking $G = \Psi_K$, we obtain the following corollary.

Proposition 8

Let F be a continuously differentiable map on a neighborhood of K and $F|_K$ be its restriction to K . Then

$$\text{(8)} \qquad \forall\, x_0 \in K\,,\quad DF|_K(x_0)(u_0) = \begin{cases} \{\nabla F(x_0)\cdot u_0\} & \text{when } u_0 \in D_K(x_0) \\ \emptyset & \text{when } u_0 \notin D_K(x_0) \end{cases}$$ ∎

Variational principle.

We generalize to the case of set-valued maps the fundamental fact that the derivative of a function at a point where it achieves the minimum vanishes.

Let $P \subset Y$ be a closed convex cone defining a preorder on Y (by calling non-negative elements those elements of P). Let F be a set-valued map from $K \subset X$ to Y. We say that $x_0 \in K$ achieves the minimum of F on K at $y_0 \in F(x_0)$ if

$$\text{(9)} \qquad \forall\, x \in K\,,\quad F(x) \subset y_0 + P\,.$$

Proposition 9

Let us assume that $x_0 \in K$ achieves the minimum of F on K at y_0 . Then,

$$\text{(10)} \qquad \forall\, u_0 \in X\,,\quad DF(x_0,y_0)(u_0) \subset P$$ ∎

Proof.

Let $v_0 \in DF(x_0,y_0)(u_0)$. For all $\varepsilon > 0$ and $\alpha > 0$, there exists $u \in u_0 + \varepsilon B$ such that

$$v_0 \in \frac{F(x_0+hu) - y_0}{h} + \varepsilon B \subset P + \varepsilon B$$

by (9). Hence $v_0 \in c\ell(P) = P$. ∎

Remark.

This inclusion is trivial when $u_0 \notin \mathrm{Dom}\, DF(x_0,y_0)$, since in this case $DF(x_0,y_0)(u_0) = \emptyset$. ∎

The following property will play an important role for defining upper contingent derivatives of a real-valued function.

Let P be a closed convex cone of Y. For any subset L, we set $L_+ = L + P$; we say that L is "comprehensive" if $L_+ = L$.

Proposition 10

Let F be a set valued map from K to Y. Then, for any $(x_0, y_0) \in \text{graph}(F)$, we have

(11) $$DF(x_0, y_0)_+ \subset DF_+(x_0, y_0).$$

If the images of F are comprehensive, the images of $DF(x_0, y_0)$ are also comprehensive. ▪

Proof.

Let $v_0 \in DF(x_0, y_0)(u_0)$ and $z \in P$. Let $\varepsilon > 0$ and $\alpha > 0$. We know that there exist $u \in u_0 + \varepsilon B$ and $h \leq \alpha$ such that $v_0 \in \dfrac{F(x_0 + hu) - y_0}{h} + \varepsilon B$. Hence

$v_0 + z \in \dfrac{F(x_0 + hu) + hz - y_0}{h} + \varepsilon B \subset \dfrac{F_+(x_0 + hu) - y_0}{h} + \varepsilon B$, and thus, $v_0 + z \in DF_+(x_0, y_0)(u_0)$. ▪

5. Upper contingent derivative of a real-valued function.

We associate with the function $V: X \to]\infty, +\infty]$ the set-valued map V_+ defined by $V_+(x) = V(x) + \mathbb{R}_+$ when $V(x) < +\infty$ and $V_+(x) = \emptyset$ when $V(x) = +\infty$. Its domain is the domain of V and its graph is the epigraph of V. We consider its contingent derivative $DV_+(x,V(x))$, which has comprehensive values by Proposition 4.10. Therefore, for all $u_0 \in X$, $DV_+(x,V(x))$ is either $\mathbb{R}$, or a half line $[v_0,\infty[$, or empty. We set

(1) $$D_+V(x)(u) \doteq \inf\{v \mid v \in DV_+(x,V(x))(u)\} .$$

It is equal to $-\infty$ if $DV_+(x,V(x)) = \mathbb{R}$, to v_0 if $DV_+(x,V(x))(u) = [v_0,\infty[$ and to $+\infty$ if $DV_+(x,V(x))(u) = \emptyset$.

Definition 1

We shall say that $D_+V(x)(u)$ is the "upper contingent derivative" of V at x in the direction u. ∎

Remark.

We can define as well $V_-(x) = V(x) - \mathbb{R}_+$ and $D_-V(x)(u) \doteq \sup\{v \mid v \in DV_-(x,V(x))(u)\}$. We say that $D_-V(x)(u)$ is the lower contingent derivative of V at x in the direction u. ∎

We begin by computing upper contingent derivatives.

Theorem 1

If V is a real-valued function, then

(2) $$D_+V(x_0)(u_0) = \liminf_{\substack{h \to 0+ \\ u \to u_0}} \frac{V(x_0 + hu) - V(x_0)}{h}$$ ∎

Proof.

Indeed, let $v_0 \in DV_+(x_0,V(x_0))(u_0)$; then, $\forall \varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\forall \alpha > 0$, there exist $u \in u_0 + \varepsilon_2 B$ and $h < \alpha$ such that $v_0 \in \dfrac{V_+(x_0 + hu) - V(x_0)}{h} + \varepsilon_1 B$. This implies that $v_0 \geq \dfrac{V(x_0 + hu) - V(x_0)}{h} - \varepsilon_1 \geq \inf\limits_{h \leq \alpha} \inf\limits_{\|u-u_0\| \leq \varepsilon_2} \dfrac{V(x_0 + hu) - V(x_0)}{h} - \varepsilon_1$. Therefore

$$v_0 \geq \liminf_{\substack{h \to 0+ \\ u \to u_0}} \frac{V(x_0 + hu) - V(x_0)}{h} - \varepsilon_1$$. Let us set for the time

$$a \doteq \liminf_{\substack{h \to 0+ \\ u \to u_0}} \frac{V(x_0 + hu) - V(x_0)}{h} .$$

So, we have proved that $a \leq D_+V(x_0)(u_0)$. On the other hand, we know that for any $M > a$, there exists $\delta > 0$ such that

$$\sup_{\substack{\alpha>0 \\ \delta>0}} \inf_{h\leq\alpha} \inf_{\|u_0-u\|\leq\delta} \frac{V(x_0 + hu) - V(x_0)}{h} \leq M .$$

This shows that $M > a$, $\forall\delta > 0$, there exist $h < \alpha$, and $u \in u_0 + \delta B$ such that:

$$\frac{V(x_0 + hu) - V(x_0)}{h} \leq M .$$

Hence $M \in \dfrac{V_+(x_0 + hu) - V(x_0)}{h}$. This proves that $a \in DV_+(x_0, V(x_0))(u_0)$. Since it is smaller than all the other ones, we infer that $a = D_+V(x_0)(u_0)$. ∎

<u>Proposition 1</u>

If the function V is locally Lipschitzean, we have

$$D_+V(x_0)(v_0) = \liminf_{h \to 0+} \frac{V(x_0 + hu_0) - V(x_0')}{h} . \tag{3}$$ ∎

<u>Proof</u>.

It is a consequence of <u>Proposition 3.2</u>, since in this case the set-valued map V_+ is upper locally Lipschitzean. One can see it directly for in this case,

$$\liminf_{h \to 0+} \frac{V(x_0 + hu_0) - V(x_0)}{h} = \liminf_{\substack{h \to 0+ \\ u \to u_0}} \frac{V(x_0 + hu) - V(x_0)}{h} .$$ ∎

So, in this case, the upper contingent derivative coincides with one of the Dini derivatives.

<u>Remark</u>

We can compute in the same way the lower contingent derivative of V : we obtain

$$D_-V(x_0)(u_0) = \limsup_{\substack{h \to 0+ \\ u \to u_0}} \frac{V(x_0 + hu) - V(x_0)}{h} . \tag{4}$$

Therefore, we always have

(5) $$D_+V(x_0)(u_0) \leq D_-V(x_0)(u_0) .$$

We shall say that the interval-valued map:

$$u_0 \mapsto [D_+V(x_0)(u_0),\ D_-V(x_0)(u_0)]$$

is the contingent gap map.

Let us mention also that

(6) $$D_+V(x_0)(-u_0) = \limsup_{\substack{h \to 0+ \\ u \to u_0}} \frac{V(x_0) - V(x_0 - hu)}{h} .$$ ■

Remark.

Let V be a function from X to $]-\infty,+\infty]$. We set

(7) $$K \doteq \{x \in X \text{ such that } V(x) \leq c\}.$$

We can characterize the contingent cone to K at x in the following way:

Proposition 2

If $V(x) = c$, then

(8) $$D_K(x) \subset \{v \mid D_+V(x)(v) \leq 0\} .$$

Proof.

If $v_0 \in D_K(x)$, then $\forall\ \varepsilon > 0$, $\forall\ \alpha > 0$, there exist $h < \alpha$ and $v \in v_0 + \varepsilon B$ such that $x + hv \in K$, i.e., such that $\frac{V(x + hv) - V(x)}{h} \leq 0$. This implies that $D_+V(x)(v_0) \leq 0$. ■

Remark.

The indicator $\phi_K : X \to]-\infty,+\infty]$ of a subset $K \subset X$ is the function defined by $\phi_K(x) = 0$ when $x \in K$ and $\phi_K(x) = +\infty$ when $x \notin K$.

Proposition 3

The upper contingent derivative of the indicator ϕ_K of $K \subset X$ is the indicator of the contingent cone $D_K(x)$:

(9) $$D_+\phi_K(x)(\cdot) = \phi_{D_K(x)}(\cdot) .$$ ■

Proof. It follows from Proposition 3.1, for $\phi_{K+} = \psi_K$ is the indicator of K . ■

6. Calculus on upper contingent derivative

The upper contingent derivative inherits the properties of the contingent derivatives of set-valued maps. In the above corollaries, we use the fact that $a \leq b$ if and only if $[b,\infty[\subset [a,\infty[$ and that $\bigcup_i [a_i,\infty[= [\min_i a_i,\infty[$.

We begin by the variational principle.

Proposition 1

Let V be a function from K to $\mathbb{R}$. If $x_0 \in K$ minimizes V on K, then

$$(1) \qquad \forall u_0 \in X, \quad 0 \leq D_+V(x_0)(u_0). \qquad \blacksquare$$

Proof.

We apply Proposition 4.9 with $F(x) = V_+(x)$, $P = \mathbb{R}_+$ and $y_0 = V(x_0)$. ■

Proposition 2

Let V be a function from K to $\mathbb{R}$ and $L \subset K$. Let $V|_L$ be the restriction of V to L. Then

$$(2) \qquad \forall x_0 \in L, \quad \forall v_0 \in D_L(x_0), \quad D_+V(x_0)(u_0) \leq D_+V|_L(x_0)(u_0).$$

Proof.

It follows from Proposition 4.6 with $F(x) \doteq V_+(x)$.

We estimate now the upper contingent derivative of the sum of two functions.

Proposition 3.

Let V and W be two functions from K to $\mathbb{R}$ and $U \doteq V + W$. Let $L \subset K$ be a subset of K. Then

$$(3)\begin{cases} \text{i)} & D_+ U(x_0)(u_0) \leq D_+V(x_0)(u_0) + D_-W(x_0)(u_0) \\ \text{ii)} & \forall u_0 \in D_L(x_0), \quad D_+V|_L(x_0)(u_0) \leq D_-V(x_0)(u_0). \end{cases}$$

Therefore, when $D_+V(x_0) = D_-V(x_0)$ (which is the case when V is convex continuous or continuously differentiable), we obtain the formula

$$(4) \qquad \forall u_0 \in D_L(x_0), \quad D_+V|_L(x_0)(u_0) = D_+V(x_0)(u_0). \qquad \blacksquare$$

Proof.

Inequality (3) i) follows from the fact that

$$\liminf_{\substack{h \to 0+ \\ v \to v_0}} (f(h,v) + g(h;v)) \leq \liminf_{\substack{h \to 0+ \\ v \to v_0}} f(h,v) + \limsup_{\substack{h \to 0+ \\ v \to v_0}} g(h,v)$$

where we set $f(h,v) = \dfrac{V(x_0 + hv) - V(x_0)}{h}$ and $g(h,v) = \dfrac{W(x_0 + hv) - W(x_0)}{h}$. We deduce inequality (3) ii) by taking for function W the indicator $\phi_L(\cdot)$ of L. So, equality (4) follows from (2) and (3) when $D_+V(x_0)(u_0) = D_-V(x_0)(u_0)$. ∎

Remark

We deduce from Proposition 4.7 that when W is continuously differentiable, we have equality

$$D_+U(x_0)(u_0) = D_+V(x_0)(u_0) + \langle \nabla W(x_0), u_0 \rangle \ . \quad \blacksquare$$

We shall now prove the chain rule formulas.

Proposition 4

Let $V : K \to \mathbb{R}$ be a function and φ be a continuously differentiable non-decreasing function from an open neighborhood of $V(K)$ to $\mathbb{R}$. Then

(5) $$D_+(\varphi V)(x_0)(u_0) \leq \varphi'(V(x_0))\, D_+V(x_0)(v_0) \quad \blacksquare$$

Proof.

Since φ is non-decreasing, $\varphi(V_+(x)) = (\varphi V)_+(x)$. Hence, we apply Proposition 4.1: with $A = \varphi$, $F(x) = V_+(x)$ and $y = V(x)$. We obtain $\varphi'(V(x_0))DV_+(x_0,V(x_0))(u_0) \subset D(\varphi V)_+(x_0,\varphi V(x_0))(u_0)$, i.e., (5).

Proposition 5.

Let V be a function from K to $\mathbb{R}$ and A a continuously differentiable map from an open subset $\Omega \subset Z$ to K. Then

(6) $$D_+V(Ax_0)(\nabla A(x_0)u_0) \leq D_+(VA)(x_0)(u_0). \quad \blacksquare$$

Proof.

We apply Proposition 4.2 with $F(x) = V_+(x)$. ∎

The following formula on the contingent derivative of the pointwise minimum of a finite number of real valued-functions is very useful.

Proposition 6

Let us consider n functions V_i from K to $\mathbb{R}$ $(i = 1,\ldots,n)$. We set $V(x) \doteq \min\limits_{i=1,\ldots,n} V_i(x)$ and $I(x) \doteq \{i \mid V_i(x) = V(x)\}$. Then

(7) $$D_+V(x_0)(u_0) = \min_{i\in I(x)} D_+V_i(x_0)(u_0) \ . \quad \blacksquare$$

Proof.

We note that $V_+(x) = \bigcup_{i\in I} V_{i+}(x)$ and that $I(x,V(x)) = I(x)$. We apply Proposition 4.3.

Proposition 7

Let us consider n functions V_i from K to $\mathbb{R}$. We set

$$(8)\quad \begin{cases} \text{i)} & W(x) \doteq \max_i V_i(x) \\ \text{ii)} & J(x_0) \doteq \{i = 1,\ldots,n \mid (x_0, W(x_0)) \notin \text{Int}\ \mathcal{E}p(V_i)\} . \end{cases}$$

Then

$$(9)\quad D_+W(x_0)(v) \geq \max_{i\in J(x_0)} D_+ V_i(x_0)(v) . \qquad \blacksquare$$

Proof.

We note that $W_+(x) = \bigcap_{j=1}^{n} V_{j+}(x)$ and that $J(x_0) = J(x_0,W(x_0))$. Then Proposition 4.4 implies that $D\,W_+(x_0,W(x_0))(v) \subset \bigcap_{i\in J(x_0,w(x_0))} DV_{i+}(x_0,W(x_0))(v)$. This inclusion implies inequality (9). $\blacksquare$

Remark

If $x_0 \in \bigcap_{i=1}^{n}$ Int Dom V_i, we note that

$$(10)\quad J(x_0) = \{i = 1,\ldots,n \mid W(x_0) = V_i(x_0)\} . \qquad \blacksquare$$

We shall study now the chain rule for the composition of a function V from X to $\mathbb{R}$ and an absolutely continuous function $t \to x(t)$. We recall that almost all t is a Lebesgue point, i.e., satisfies $x'(t_0) = \lim_{h\to 0} \frac{1}{h}\int_{t_0}^{t_0+h} x'(\tau)d\tau$.

Proposition 8

Let $x(\cdot)$ be an absolutely continuous function from $[t_0-\eta, t_0+\eta]$ to $K \subset X$ and assume that

$$(11)\quad x'(t_0) = \lim_{h\to 0+} \frac{1}{h}\int_{t_0}^{t_0+h} x'(\tau)d\tau .$$

(This limit belongs to $D_K(x(t_0))$.)

Set $v(t) \doteq V(x(t))$. Inequality

$$(12)\quad D_+V(x(t_0))(x'(t_0)) \leq \limsup_{h\to 0+} \frac{v(t_0+h) - v(t_0)}{h}$$

always holds true.

Moreover, if V is the restriction to K of a locally Lipschitzean function $\tilde{V}$ defined on a neighborhood of K, then

$$\liminf_{h \to 0+} \frac{v(t_0+h) - v(t_0)}{h} \leq D_+V(x(t_0))(x'(t_0)). \tag{13}$$ ∎

Therefore, if we know that $v'(t_0) = \lim_{h \to 0+} \frac{v(t_0+h) - V(t_0)}{h}$ exists, we get

$$v'(t_0) = D_+V(x(t_0))(x'(t_0)) . \tag{14}$$

Proof.

a). We set $v_h = \frac{1}{h}\int_0^h x'(\tau)d\tau$. So, we can associate to any $\varepsilon > 0$ a positive number $\beta > 0$ such that

$$\forall\, h \leq \beta, \quad \|v_h - x'(t_0)\| \leq \varepsilon. \tag{15}$$

We observe that $x(t_0 + h) = x(t_0) + h\, v_h \in K$. Hence $x'(t_0) \in D_K(x(t_0))$. We set

$$v'_{\#}(t_0) \doteq \inf_{\alpha>0} \sup_{h<\alpha} \frac{v(t_0 + h) - v(t_0)}{h}.$$

We have

$$D_+V(x(t_0))(x'(t_0)) = \liminf_{\substack{h \to 0+ \\ v \to v_0}} \frac{V(x(t_0) + h\, v) - V(x(t_0))}{h}$$

$$\leq \sup_{\varepsilon>0} \inf_{\alpha>0} \sup_{h<\alpha} \inf_{v \in v_0+\varepsilon B} \frac{V(x(t_0) + hv) - V(x(t_0))}{h}$$

$$\leq \sup_{\varepsilon>0} \inf_{\alpha>0} \sup_{h<\alpha} \frac{V(x(t_0) + hv_h) - V(x(t_0))}{h} = v'_{\#}(t_0) .$$

b). Let $v'_b(t_0) \doteq \liminf_{h \to 0+} \frac{v(t_0 + h) - V(t_0)}{h}$. Then, for all $\varepsilon > 0$, there exists $\gamma > 0$ such that, $\forall\, h \leq \gamma$, $v'_b(t_0) \leq \frac{v(t_0 + h) - v(t_0)}{h} + \varepsilon$. Let ℓ be the Lipschitz constant of V at $x(t_0)$. Then, for all $v \in x'(t_0) + \varepsilon B$ and $h \leq \alpha \doteq \min(\beta,\gamma)$, we have, thanks to (15)

$$v'_b(t_0) \leq \frac{V(x(t_0) + hv_h) - V(x(t_0))}{h} + \varepsilon \leq \frac{V(x(t_0) + hv) - V(x(t_0))}{h} + 2\ell\varepsilon .$$

On the other hand, we know that there exists $h \leq \alpha$ and $v \in x'(t_0) + \varepsilon B$ such that

$$\frac{\tilde V(x(t_0)+hv)-\tilde V(x(t_0))}{h} \leq D_+\tilde V(x(t_0))(x'(t_0)) + \varepsilon.$$ Hence

$$v_b'(t_0) \leq D_+\tilde V(x(t_0))(x'(t_0)) + (2\ell+1)\,\varepsilon.$$

By letting $\varepsilon \to 0$, we obtain $v_b'(t_0) \leq D_+\tilde V(x(t_0)(x'(t_0))$. Since $x'(t_0) \in D_K(x(t_0))$, Proposition 2 implies that $D_+\tilde V(x(t_0))(x'(t_0)) \leq D_+V(x(t_0))(x'(t_0))$. Hence (11) holds true. ∎

Remark.

If both V and x are locally Lipschitzean, then v is also locally Lipschitzean and inequality (13) can be written, by setting $D_+v(t_0) \doteq D_+v(t_0)(1)$,

(16) $$D_+v(t_0) \leq D_+V(x(t_0))(x'(t_0)).$$ ∎

We can "integrate" inequalities involving contingent derivatives.

Proposition 9.

Let v be a continuous function from $[0,T]$ to $\mathbb{R}$ and w be a bounded upper semi-continuous function from $[0,T[$ to $\mathbb{R}$ which is bounded above. We assume that

(17) $$\forall\, t \in [0,T[\,,\quad D_+v(t) + w(t) \leq 0\,.$$

Then, for all $0 \leq a < b < T$, we obtain the inequality

(18) $$v(b) - v(a) + \int_a^b w(\tau)d\tau \leq 0$$ ∎

Proof.

Let $t \in [a,b]$ and $\varepsilon > 0$ be fixed. Since w is upper semicontinuous, there exists $\eta \in]0,\varepsilon[$ such that, $\forall\, h \leq \eta$,

(19) $$\frac{1}{h}\int_t^{t+h} w(\tau)d\tau < w(t) + \varepsilon/2$$

and there exists $h_t \leq \eta$ and a_t such that $|a_t - 1| \leq \varepsilon$ that satisfy

(20) $$\frac{v(t+h_ta_t)-v(t)}{h_t} < \liminf_{\substack{h\to 0+\\ a\to 1}} \frac{v(t+ha)-v(t)}{h} + \varepsilon/2\,.$$

Hence t belongs to the subset

(21) $$N(t) \doteq \{s \in [a,b] \,|\, v(s+h_ta_t) - v(s) + \int_s^{s+h_t} w(\tau)d\tau < \varepsilon h_t\}$$

which is open since v is continuous. Let us set

$$m = \sup_{t\in[a,b]} w(t) < +\infty\,.$$

Hence the compact interval $[a,b]$ can be covered by n open subsets $N(t_i)$. We set $a_i \doteq a_{t_i}$, $h_i \doteq h_{t_i}$ and $h_0 = \min_{i=1,\dots,n} h_i > 0$.

We construct by induction the following sequence. We set $\tau_0 \doteq a$; it belongs to some $N(t_{i_1})$ and thus, by taking $s \doteq \tau_0 (=a)$ and $\tau_1 = h_{i_1} a_{i_1}$, we obtain

$$(22) \qquad v(\tau_1) - v(\tau_0) + \int_{\tau_0}^{\tau_1} w(\tau)d\tau \leq \varepsilon h_1 + \int_{h_1+a}^{h_1 a_1 + a} w(\tau)d\tau.$$

Assume that for $j \leq k$, we have constructed a sequence $\tau_j \in [a,b[$ $(1 \leq j \leq k)$ such that

$$(23) \qquad v(\tau_j) - v(\tau_{j-1}) + \int_{\tau_{j-1}}^{\tau_j} w(\tau)d\tau \leq \varepsilon h_j + \int_{h_j+\tau_{j-1}}^{h_j a_j + \tau_{j-1}} w(\tau)d\tau .$$

Then τ_k belongs to some $N(t_{i_{k+1}})$; we set

$$s \doteq \tau_k \text{ and } \tau_{k+1} \doteq \tau_k + h_{i_{k+1}} a_{i_{k+1}}$$

and we deduce that

$$(24) \qquad v(\tau_{k+1}) - v(\tau_k) + \int_{\tau_k}^{\tau_{k+1}} w(\tau)d\tau \leq \varepsilon h_{i_{k+1}} + \int_{h_{i_{k+1}}+\tau_{k-1}}^{h_{i_{k+1}} a_{i_{k+1}} + \tau_{k-1}} w(\tau)d\tau .$$

If $\tau_k < b \leq \tau_{k+1}$, we stop the construction. Otherwise, we continue. Since $\tau_{k+1} - \tau_k = a_{i_{k+1}} h_{i_{k+1}} \geq h_{0/2} > 0$, we are sure that eventually, after a finite number of steps, we shall have an index k such that $\tau_k < b \leq \tau_{k+1}$.

By adding the above inequalities from $j = 1$ to k, we obtain

$$(25) \qquad v(\tau_{k+1}) - v(a) + \int_a^{\tau_{k+1}} w(\tau)d\tau \leq \varepsilon\Big(\sum_{i=1}^{k+1} h_i\Big) m' \text{ where } m' \text{ is a constant.}$$

When ε converges to 0, τ_k and τ_{k+1} converge to b (for $\tau_{k+1} - \tau_k = h_{i_{k+1}} a_k \leq \eta(1+\varepsilon) \leq \varepsilon(1+\varepsilon)$) and thus, we deduce that

$$v(b) - v(a) + \int_a^b w(\tau)d\tau \leq 0 . \qquad \blacksquare$$

In particular, we obtain the following useful consequence.

<u>Proposition</u> 10

Let v be a continuous function from $[0,T]$ to $\mathbb{R}$ satisfying

$$(24) \qquad \forall\, t \in \,]0,T[\;,\; D_+ v(t) \leq 0 .$$

Then the function v is non-increasing. $\blacksquare$

7. Contingent derivatives of marginal functions and marginal maps.

Let us consider a family of minimization problems depending upon a parameter y: Minimize the function $x \mapsto U(x,y)$ on a subset $F(y)$. We define the marginal function V by

$$(1) \qquad V(y) \doteq \inf_{x \in F(y)} U(x,y)$$

and the marginal map G by

$$(2) \qquad G(y) \doteq \{x \in F(y) \mid U(x,y) = V(y)\} .$$

Sensibility analysis deals with the behavior of the marginal functions V and the marginal G when the parameter y varies around a fixed value y_0. This is of upmost relevance in economics, for instance, as well as in other fields. In the convex case, we refer to Rockafellar [1]. In the locally Lipschitzean case, to Aubin-Clarke [3] and to Aubin [4]. We shall study in this section the properties of the contingent derivatives of the marginal function V and the marginal map G.

For simplicity, we assume that

$$(3) \quad \begin{cases} \text{i)} & F \text{ is a compact-valued map from a subset } M \text{ of a Hilbert space } Y \\ & \text{to a Hilbert space } X \\ \text{ii)} & \forall y \in M ,\ x \to U(x,y) \text{ is lower semicontinuous.} \end{cases}$$

Hence the marginal map G is well defined on M.

Proposition 1

Let $y_0 \in K$ and $x_0 \in G(y_0)$ achieve the minimum of $U(\cdot,y_0)$ on $F(y_0)$. Then,

$$(4) \quad \forall v_0 \in \operatorname{Dom} DF(y_0,x_0),\ \forall u_0 \in DF(y_0,x_0)(v_0),\ D_+V(y_0)(v_0) \leq D_-U(x_0,y_0)(u_0,v_0).$$

Furthermore,

$$(5) \quad \forall v_0 \in \operatorname{Dom} DG(y_0,x_0),\ \forall u_0 \in DG(y_0,x_0)(v_0),\ D_+U(x_0,y_0)(u_0,v_0) \leq D_-V(y_0)(v_0). \quad \blacksquare$$

Proof.

a). Let $u_0 \in DF(y_0,x_0)(v_0)$. Then, for all $\varepsilon > 0$, $\alpha > 0$, there exist $h < \alpha$, $v \in v_0 + \varepsilon B$, $u \in u_0 + \varepsilon B$ such that $x_0 + hu \in F(y_0 + hv)$, and therefore, such that $V(y_0 + hv) \leq U(x_0 + hu, y_0 + hv)$. Since $V(y_0) = U(x_0,y_0)$, we deduce that

$$\frac{V(y_0 + hu) - V(y_0)}{h} \leq \frac{U(x_0 + hu, y_0 + hv) - U(x_0,y_0)}{h} .$$

This implies inequality (4).

b). Let $u_0 \in DG(y_0,x_0)(v_0)$. Then, for all $\varepsilon > 0$, $\alpha > 0$, there exist $h < \alpha$, $v \in v_0 + \varepsilon B$, $u \in u_0 + \varepsilon B$ such that $x_0 + hu \in G(y_0 + hv)$. *Hence* $V(y_0 + hv) = U(x_0 + hu, y_0 + hv)$. Since $V(y_0) = U(x_0,y_0)$, we deduce that

$$\frac{U(x_0 + hu,\ y_0 + hv) - U(x_0,y_0)}{h} = \frac{V(y_0 + hu) - V(y_0)}{h}$$

This implies inequality (5). ∎

8. Ekeland's variational principle

We shall derive the approximate variational principle of Ekeland in the following form:

Theorem 1

Let $K \subset X$ be a closed subset of a Hilbert space and $V: K \to [0,\infty[$ be a lower semicontinuous function. Then we can associate with any $\varepsilon > 0$ and any $x_\varepsilon \in K$ satisfying $V(x_\varepsilon) \leq \inf_{x \in K} V(x) + \varepsilon^2$ an element $\bar{x}_\varepsilon \in K$ which satisfies

$$(1)\begin{cases} \text{i)} & \|x_\varepsilon - \bar{x}_\varepsilon\| \leq \varepsilon \\ \text{ii)} & \forall u \in X, \quad 0 \leq D_+V(\bar{x}_\varepsilon)(u) + \varepsilon\|u\| . \end{cases}$$

Proof.

We derive this result from Ekeland's variational principle (see Ekeland [1] or Aubin [1], p. 174.)

Theorem (Ekeland)

Let K be a closed subset of a Hilbert space and V be a lower semicontinuous function from K to $[0,\infty[$. Then we can associate with any $\varepsilon > 0$ and any $x_\varepsilon \in K$ satisfying $V(x_\varepsilon) \leq \inf_{x \in K} V(x) + \varepsilon^2$ an element $\bar{x}_\varepsilon \in K$ which satisfies $\|x_\varepsilon - \bar{x}_\varepsilon\| \leq \varepsilon$ and $V(\bar{x}_\varepsilon) = \min_{x \in K} [V(x) + \varepsilon\|x - \bar{x}_\varepsilon\|]$.

Let $u \in \operatorname{Dom} D_+V(\bar{x}_\varepsilon)$. Then, for any $\eta > 0$, $\delta > 0$, $\alpha > 0$, $\exists h \leq \alpha$, $\exists v \in u + \delta B$ such that

$$\frac{V(\bar{x}_\varepsilon + hv) - V(\bar{x}_\varepsilon)}{h} \leq D_+V(\bar{x}_\varepsilon)(u) + \eta .$$

By Ekland's variational principle, we have

$$-\varepsilon\delta - \varepsilon\|u\| \leq -\varepsilon\|v\| \leq \frac{V(\bar{x}_\varepsilon + h\,v) - V(\bar{x}_\varepsilon)}{h}$$

Therefore, we infer that

$$0 \leq D_+V(\bar{x}_\varepsilon)(u) + \varepsilon\|u\| + \varepsilon\delta + \eta$$

By letting δ and η converge to 0, we obtain the desired inequality.

9. Surjectivity theorems.

We devote this section to the generalization to the case of upper semicontinuous maps with compact values of the inverse function theorem. We shall begin by proving theorems of existence of stationary points, then deduce surjectivity theorems and we shall end with a theorem insuring that the image by F of a neighborhood of $\bar{x}$ is a neighborhood of $F(\bar{x})$. These theorems, due to Ekeland, are simple consequences of his variational principle.

It is convenient to start with the following lemma.

Lemma 1

Let G be a set-valued map from $K \subset X$ to $L \subset Y$ and V be a continuous convex function defined on a neighborhood of L. We define the set-valued map $H \doteq V(G)$ from K to $\mathbb{R}$ by

$$(1) \qquad \forall x \in K, \quad H(x) = \{V(y)\}_{y \in G(x)} .$$

Assume that

$$(2) \qquad \begin{cases} \exists x_0 \in K, \quad \exists y_0 \in G(x_0) \text{ and } a_0 \in \mathbb{R} \text{ such that} \\ \forall x \in K, \quad H(x) \subset V(y_0) + a_0\|x - x_0\| + \mathbb{R}_+ . \end{cases}$$

Let $DV(y)(\cdot)$ denote the derivative of V at y. Then

$$(3) \qquad \forall u_0 \in X, \quad \forall v_0 \in DG(x_0,y_0)(u_0), \quad a_0\|u_0\| \le DV(y_0)(v_0)$$ ■

Proof.

Let $v_0 \in DG(x_0,y_0)(u_0)$. Hence, for all $\alpha,\beta,\gamma > 0$, there exist $h \le \alpha$, $u \in u_0 + \gamma B$ such that

$$v_0 \in \frac{G(x_0 + hu) - y_0}{h} + \beta B .$$

So, we can write $y_0 + hv_0 = y_h + \beta hb$ where $y_h \in G(x_0+hu)$ and $b \in B$.

Since V is convex, we deduce that

$$(4) \qquad V(y_h) - V(y_h - hv_0) \le h\, DV(y_h)(v_0)$$

and since V is continuous (and thus, locally Lipschitzean), there exists $\ell > 0$ such that

$$V(y_h - hv_0) - V(y_0) \le \ell\|y_h - y_0 - hv_0\| \le \ell\beta h .$$

We recall that $y \mapsto DV(y)(v_0)$ is upper semicontinuous. Then for all $\varepsilon > 0$, $\exists \eta > 0$ such that

(6) $$DV(y_h)(v_0) \leq DV(y_0)(v_0) + \varepsilon \quad \text{when} \quad \|y_h - y_0\| \leq \eta.$$

By taking $\alpha \leq \eta/(\beta + \|v_0\|)$ if necessary, inequalities (4), (5) and (6) imply that

$$V(y_h) - V(y_0) \leq h(DV(y_0)(v_0) + \varepsilon + \ell\beta)$$

or,

$$DV(y_0)(v_0) \in \frac{H(x_0 + hu\ ) - V(y_0)}{h} + \mathbb{R}_+ + (\varepsilon + \ell\beta)B \quad .$$

We use now assumption (2): we obtain

$$DV(y_0)(v_0) \in a_0\|u\ \| + (\varepsilon + \ell\beta)B + \mathbb{R}_+ \quad .$$

By letting ε, β and γ go to 0, we infer that

$$DV(y_0)(v_0) \geq a_0\|u_0\| . \qquad \blacksquare$$

When L is a subset of X, we set

$$m(L) = \{x \in L \text{ such that } \|x\| = \min_{y \in L} \|y\|\}.$$

<u>Theorem 1</u>

Let F be an upper semicontinuous map with compact values from a compact subset K of a Hilbert space X to a Hilbert space Y. We assume that

(7) $$\forall x \in K, \ \exists y \in m(F(x)), \ \exists u \in X \text{ such that } -y \in DF(x,y)(u) .$$

Then there exists a stationary point $\bar{x} \in K$ of F. $\blacksquare$

<u>Proof</u>.

Since the function $x \mapsto \|m(F(x))\|$ is lower semicontinuous (for F is upper semicontinuous with compact values) and since K is compact, there exists $x_0 \in K$ which achieves the minimum of $x \to \|m(F(x))\|$ on K. Let us choose $y_0 \in m(F(x_0))$. We set $V(x) = \|x\|$ and $H(x) = \{V(y)\}_{y \in F(x)}$. It is clear that

$$\forall x \in K, \quad H(x) \subset V(y_0) + \mathbb{R}_+$$

since, if $c \in H(x)$, then $c \geq \|m(F(x))\|^2 \geq \|m(F(x_0))\| = V(y_0)$. So, we apply <u>Lemma 1</u> with $a_0 = 0$. We deduce that

$$\forall u_0 \in X, \ \forall v_0 \in DF(x_0,y_0)(u_0), \quad 0 \leq DV(y_0)(v_0) .$$

By assumption (7), we can take $v_0 = -y_0$; since

$$DV(y_0)(-y_0) = \lim_{h \to 0+} \frac{\|y_0 - hy_0\| - \|y_0\|}{h} = -\|y_0\| ,$$

we deduce that $\|y_0\| \leq 0$. Hence $y_0 = 0 \in F(x_0)$. ■

Corollary 1

Let F be a Gâteaux differentiable continuous map from a neighborhood of a compact subset K to Y . Assume that

(8) $\forall x \in K , \quad \exists u \in D_K(x)$ such that $\nabla F(x)u = -F(x)$.

Then there exists a stationary point $x_0 \in K$ of F . ■

By using Ekeland's theorem, we can replace the compactness assumption on K in Theorem 1 by another assumption on the growth of the inverse of the contingent derative.

Theorem 2 (Ekeland)

Let F be an upper semicontinuous map with compact values from a closed subset K of a Hilbert space X to a Hilbert space Y . We assume that

(9) $\begin{cases} \exists c > 0 \text{ such that, } \forall x \in K , \quad \exists y \in m(F(x)), \quad \exists u \in X \\ \textit{such that } -y \in DF(x,y)(u) \textit{ and } c\|u\| \leq \|y\| . \end{cases}$

Then there exists a stationary point $\bar{x} \in K$ *of* F . ■

Proof.

By Ekeland's theorem, we can associate with any $\varepsilon < c$ an element $x_0 \in K$ such that, for all $x \in K$,

(10) $\|m(F(x_0))\| \leq \|m(F(x))\| + \varepsilon\|x - x_0\|$.

If $m(F(x_0)) \doteq y_0 = 0$, the theorem is proved. Otherwise we take $V(x) \doteq \|x\|$ and $H(x) = \{V(y)\}_{y \in F(x)}$. Inequality (10) can be written

(11) $\forall x \in K , \quad H(x) \subset V(y_0) - \varepsilon\|x - x_0\| + \mathbb{R}_+$.

Hence we apply Lemma 1 with $a_0 = -\varepsilon$. We obtain: $\forall u_0 \in X$, $\forall v_0 \in DF(x_0,y_0)(u_0)$, $-\varepsilon\|u_0\| \leq DV(y_0)(v_0)$. By assumption (9), we can take $v_0 = -y_0$ and $u_0 \in X$ satisfying $c\|u_0\| \leq \|y_0\|$. Since $DV(y_0)(-y_0) = -\|y_0\|$, we obtain the contradiction $-\varepsilon\|u_0\| \leq -\|y_0\| \leq -c\|u_0\|$. So $y_0 = 0$. ■

Corollary 2

Let F be a Gâteaux-differentiable continuous map from a neighborhood of a closed subset $K \subset X$ to Y. Assume that

$$(12) \quad \begin{cases} \exists c > 0 \text{ such that, } \forall x \in K, \quad \exists u \in D_K(x) \text{ such that} \\ \nabla F(x)u = -F(x) \text{ and } c\|u\| \leq \|F(x)\|. \end{cases}$$

then there exists a stationary point $x_0 \in K$ of F. ∎

In particular, we can take $K = X$. We obtain :

Corollary 3

Let F be a Gâteaux-differentiable continuous map from X to Y. Assume that

$$(13) \quad \begin{cases} \exists c > 0 \text{ such that, } \forall x \in X, \quad \exists u \in X \text{ satisfying} \\ \nabla F(x)u = -F(x) \text{ and } c\|u\| \leq \|F(x)\|. \end{cases}$$

Then there exists a stationary point x_0 of F. ∎

By replacing the set-valued map F by $G(x) = F(x) - y$, we obtain solutions to the inclusion $y \in F(x)$. Therefore, we obtain the following surjectivity theorems.

Theorem 3

Let F be an upper semicontinuous map with compact values from a closed subset K of a Hilbert space X to a Hilbert space Y. We assume that

$$(14) \quad \begin{cases} \exists c > 0 \text{ such that, } \forall x \in K, \ \forall y \in F(x), \ \forall v \in Y, \\ \exists u \in X \text{ satisfying } v \in DF(x,y)(u) \text{ and } c\|u\| \leq \|v\|. \end{cases}$$

Then for all $y \in Y$, there exists a solution $x \in K$ to the inclusion $y \in F(x)$ ∎

In other words, $F(K) = Y$. Theorem 3 says that F is a surjective set-valued map.

For smooth single-valued maps, we obtain the following Corollaries.

Corollary 4

Let F be a Gâteaux-differentiable continuous map from a neighborhood of a closed subset $K \subset X$ to Y. Assume that

$$(15) \quad \begin{cases} \exists c > 0 \text{ such that, } \forall x \in K, \ \forall v \in Y, \quad \exists u \in D_K(x) \\ \text{satisfying } \nabla F(x)(u) = v \text{ and } c\|u\| \leq \|v\|. \end{cases}$$

Then $F(K) = Y$. ∎

Corollary 5

Let F be a Gateaux-differentiable continuous map from X to Y. Assume that

$$(16)\quad \begin{cases} \exists\, c > 0 \text{ such that, } \forall\, x \in X\,,\ \forall\, v \in Y\,,\ \exists\, u \in X \text{ satisfying} \\ \nabla F(x)u = v \text{ and } c\|u\| \le \|v\|. \end{cases}$$

Then F is surjective. ■

We prove now an adaptation of the inverse function theorem.

Theorem 4 (Ekeland)

Let F be an upper semicontinuous map with compact values from a neighborhood U of $\bar{x} \in X$ to Y. We assume that

$$(17)\quad \begin{cases} \exists\, c > 0 \text{ such that } \forall\, x \in u\,,\ \forall\, y \in F(x)\,, \\ \forall\, v \in Y\,,\ \exists\, u \in X \text{ satisfying} \\ DF(x,y)(u) = v \text{ and } c\|u\| \le \|v\|\,. \end{cases}$$

Then $F(U)$ is a neighborhood of $F(\bar{x})$. ■

Proof

Let U contain a closed ball of center $\bar{x}$ and radius $\eta > 0$ and let $\bar{y} \in F(\bar{x})$. We claim that $F(\bar{x} + \eta B)$ contains the balls $\bar{y} + \varepsilon B$ with $\varepsilon < c\,\eta$.

Indeed, pick $y_1 \in \bar{y} + \varepsilon B$ and set $G(x) \doteq F(x) - y_1$. We apply Ekeland's theorem in the stronger form to the function $\|m(F(x) - y_1)\|$, taking ε as above. Noting that $\|m(F(\bar{x}) - y_1)\| \le \varepsilon$, we get some point x_0 such that

$$(18)\quad \begin{cases} \text{i)} & \|x_0 - \bar{x}\| \le \eta \\ \text{ii)} & \forall\, x \in \bar{x} + \eta B,\ \|m(F(x_0) - y_1)\| \le \|m(F(x) - y_1)\| + \varepsilon\eta^{-1}\|x - x_0\|. \end{cases}$$

Take $y_0 \in m(F(x_0) - y_1)$. Either it is zero (and $y_1 \in F(x_0)$) or $V(y_0) = \|m(F(x_0) - y_1)\| > 0$. In this case we obtain a contradiction. Indeed, inequality (18) ii) implies that

$$\forall\, x \in K\,,\ H(x) \subset V(y_0) - \varepsilon\eta^{-1}\|x - x_0\| + \mathbb{R}_+ \text{ (where } H(x) \doteq V(G(x)).)$$

Hence we apply Lemma 1 with $a_0 = -\varepsilon\eta^{-1}$: We get

$$\forall\, u_0 \in X\,,\ \forall\, v_0 \in DG(x_0,y_0)(u_0)\,,\ -\varepsilon\eta^{-1}\|u_0\| \le DV(y_0)(u_0)\,.$$

But $DG(x_0,y_0)(u_0) = DF(x_0,y_0 + y_1)(u_0)$. By assumption (17), we can choose $v_0 = -y_0$ and $u_0 \in X$ such that $c\|u_0\| \le \|y_0\|$. Hence, since $DV(y_0)(-y_0) = -\|y_0\|$, we obtain the contradiction $-\varepsilon\eta^{-1}\|u_0\| \le -\|y_0\| \le -c\,u_0$. So $v_0 = 0$, i.e., $y_1 \in F(x_0)$. ■

10. The Newton method

We proved in Corollary 9.1 that when F is a continuously differentiable map from a neighborhood of a compact subset $K \subset \mathbb{R}^n$ to $\mathbb{R}^m$ that satisfies the condition

(1) $\quad \forall x \in K, \quad \exists v \in D_K(x)$ such that $\nabla F(x)v = -F(x)$,

then there exists a stationary point $x_* \in K$ of F.

These assumptions imply also that there exist trajectories $x(\cdot)$ of the differential inclusion

(2) $$\nabla F(x)x' = -F(x), \quad x(0) = x_0$$

that remain in K and that converge to a stationary point of F when $t \to \infty$. (See Haddad [1].)

We can consider such trajectories as the continuous analogs of the classical Newton method, which yields the discrete trajectory defined recursively by

(3) $$\nabla F(x_n)(x_{n+1} - x_n) = -F(x_n); \quad x_0 \text{ is given.}$$

Theorem 1

Let F be a continuously differentiable map from a neighborhood of a closed subset $K \subset \mathbb{R}^n$ to $\mathbb{R}^m$ satisfying

(4) $$\begin{cases} \exists c > 0 \text{ such that, } \forall x \in K, \ \exists v \in D_K(x) \cap cB \text{ satisfying} \\ \nabla F(x)v = -F(x). \end{cases}$$

Then there exists a viable trajectory of the implicit differential equation (2) that satisfies

$$F(x(t)) = e^{-t}F(x(0)).$$

Thus the cluster points of $x(t)$ (if any) are stationary points of F. ■

Proof.

We set $G(x) \triangleq -\nabla F(x)^{-1}F(x)$. Trajectories of the differential inclusion $x' \in G(x)$ are the trajectories of the implicit differential equation (2). Assumptions of Haddad's theorem [See Haddad [1]] on differential inclusions are satisfied. Hence there exists viable trajectories of the implicit differential equation (2). Consider any such trajectory. Then, since F is continuously differentiable, we have

$$\frac{d}{dt}F(x(t)) = \nabla F(x(t))x'(t) = -F(x(t)).$$

So $y(t) \doteq F(x(t))$ is equal to $e^{-t}y(0)$. Therefore, $F(x(t))$ converges to 0 when $t \to \infty$. Any cluster point $x_* \in K$, limit of a subsequence $x(t_n)$ when $t_n \to \infty$, satisfies $F(x_*) = \lim F(x(t_n)) = 0$, and thus, is a stationary point of F. ∎

Recall that the above sufficient condition for existence of a stationary point of F can be extended to set-valued maps (see Theorem 9.1): We replace the tangential condition (4) by

$$(5) \qquad \forall x \in K, \quad \exists y \in F(x), \quad \exists u \in X \text{ such that } -y \in DF(x,y)(u)$$

where $DF(x,y)$ is the contingent derivative of the set-valued map F.

We can generalize the Newton method if we assume, for instance, that

$$(6) \begin{cases} \text{there exists a bounded upper hemicontinuous proper set-valued map } G \\ \text{from graph } (F) \text{ to the closed convex subsets of } \mathbb{R}^n \text{ such that} \\ \quad \forall x \in K, \ \forall y \in F(x), \quad \exists u \in G(x,y) \text{ such that } -y \in DF(x,y)(u). \end{cases}$$

Theorem 2 (Saint-Pierre)

Let F be a proper map from $K \subset \mathbb{R}^n$ to $\mathbb{R}^m$ with closed graph. We posit Assumption (6). Then, for any $x_0 \in K$ and $y_0 \in F(x_0)$, there exists a solution to the differential inclusion

$$(7) \qquad x'(t) \in G(x(t), e^{-t}y_0)$$

that satisfies

$$(8) \qquad \forall t \geq 0, \quad x(t) \in K \text{ and } e^{-t}y_0 \in F(x(t)).$$

Thus the cluster points of $x(t)$ (if any) are stationary points of F. ∎

Proof.

We consider the differential inclusion

$$(9) \qquad x' \in G(x,y), \quad y' = -y$$

with the initial condition $x(0) = x_0$, $y(0) = y_0$.

Condition (6) implies that

$$(10) \qquad \forall (x,y) \in \text{graph}(F), \quad (G(x,y),-y) \cap D_{\text{graph}(F)}(x,y) \neq \emptyset.$$

Then Haddad's theorem (See Haddad [1]) implies that there exists a trajectory $(x(t),y(t))$ of this differential inclusion which remains in graph (F). Furthermore, $y(t) = e^{-t}y_0$. Hence $e^{-t}y_0 \in F(x(t))$. The rest of the theorem ensues. ∎

Remark.

We note that we can devise a whole family of algorithms that converge to a stationary point of F. Let H be any map from $\mathbb{R}^n$ to itself such that

$$(11)\quad \begin{cases} \text{the solution of } y' = H(y),\ y(0) = y_0 \text{ is unique and} \\ \text{converges to } 0,\ \text{when } t \to \infty . \end{cases}$$

We associate with such a map H a bounded continuous map G (single-valued for the sake of simplicity) such that

$$(12)\qquad \forall\ (x,y) \in \operatorname{graph}(F),\ \ H(y) \in DF(x,y)(G(x,y)).$$

Then there exist solutions to the differential equation

$$(13)\qquad x'(t) \in G(x(t),y(t)),\ \ y'(t) = H(y(t)),\ x(0) = x_0\ ,\ y(0) = y_0$$

such that

$$(14)\qquad \forall\ t \geq 0\ ,\ \ y(t) \in F(x(t))\ .$$

Since $\lim_{t\to\infty} y(t) = 0$ by assumption, the cluster points of $x(t)$ (if any) are stationary points of F. ∎

11. Liapunov functions and existence of stationary points.

We shall use the existence of Liapunov function which we define below for proving the existence of stationary points. In the next sections, we shall prove that the existence of these Liapunov functions imply also that solutions to the associated differential inclusion do converge in some sense to stationary points of F when $t \to 0$.

We begin with a particular case, which is a "stationary point" version of the Aubin-Siegel fixed point theorem, which is an extension to the case of set-valued maps of the Caristi fixed point theorem (See Aubin-Siegel [1], Brézis-Browder [1] and Caristi [1]).

Theorem 1

Let K be a closed subset of the Hilbert space X, $F:K \to X$ be a set-valued map and $V:K \to [0,\infty[$ be a lower semicontinuous function satisfying

(1) $\quad \forall x \in K, \quad \exists v \in F(x)$ such that $D_+V(x)(v) + \|v\| \leq 0$.

Then there exists a stationary point $\bar{x} \in K$ of F. ■

Proof.

Take $\varepsilon < 1$ and $\bar{x}_\varepsilon \in K$ satisfying, thanks to Theorem 7.1,

$$\forall u \in X,\ 0 \leq D_+V(\bar{x}_\varepsilon)(u) + \varepsilon\|u\| .$$

By assumption, we can take $u \in F(\bar{x}_\varepsilon)$ satisfying $D_+V(\bar{x}_\varepsilon)(u) \leq -\|u\|$. Hence $(1-\varepsilon)\|u\| \leq 0$, i.e., $u = 0 \in F(\bar{x}_\varepsilon)$.

Remark.

This theorem can be regarded as the "stationary point" version of the "Caristi's fixed point theorem".

Theorem 2 (Caristi)

Let K be a closed subset, $g:K \to K$ be a single-value map and V be a lower semicontinuous map from K to $\mathbb{R}_+$. If

$$\forall x \in K, \quad V(g(x)) - V(x) + \|g(x) - x\| \leq 0 ,$$

then g has a fixed point. ■

So, we have proved that if there exists a lower semicontinuous function V such that $D_+V(x)(v) + \|v\| \leq 0$ for all $x \in K$, there exist a stationary point. The question arises whether existence of stationary points implies the existence of a function V satisfying the above condition.

Theorem 3 (Moreau)

Let K be a closed subset and F be a set-valued map from K to X satisfying

(2) $-F$ is monotone $(\forall\, x,y \in K\ ,\ \forall\, u \in F(x),\ \forall\, v \in F(y), \langle u - v, x - y \rangle \leq 0)$

We assume that the set $F^{-1}(0)$ of stationary points of F has a non-empty interior. Let us associate with any $x_0 \in \text{Int}\, F^{-1}(0)$ the function V defined by $V(x) = \frac{1}{2\rho}\|x - x_0\|^2$ where $\rho = d(x_0, \complement F^{-1}(0)) > 0$. Then

(3) $\forall\, x \in K\ ,\ \forall\, v \in F(x)\ ,\ DV(x)(v) + \|v\| \leq 0$.

and, consequently, for any $x \in x_0 + \rho\, \text{Int}\, B\ ,\ F(x) = \{0\}$. ■

Proof.

Let us take $x \in K$ and $v \in F(x)$. Since $x_0 + \rho B \subset F^{-1}(0)$, then $x_0 - \frac{\rho v}{\|v\|}$ is a stationary point of F . The monotonicity of $-F$ implies that

(4) $\langle v, x - x_0 \rangle + \rho\|v\| = \langle v - 0, x - x_0 + \frac{\rho v}{\|v\|} \rangle \leq 0$.

Let $x = x_0 + \rho u \in x_0 + \rho\, \text{Int}\, B$, and $v \in F(x)$. We infer that $\rho\langle v,u \rangle + \rho\|v\| \leq 0$. Hence $\|v\| \leq \|v\|\|u\| < \|v\|$, which is impossible when $v \neq 0$. ■

We may generalize Theorem 1 by introducing the concept of Liapunov function V with respect to a set-valued map F and a given function W defined on graph (F)

Definition 1

Let F be a set-valued map from $K \subset X$ to X and W be a function defined on graph (F). We shall say that the function V defined on K is a Liapunov function for F with respect to W if it satisfies the following "Liapunov property"

(5) $\forall\, x \in K\ ,\ \exists\, v \in F(x)$ such that $D_+V(x)(v) + W(x,v) \leq 0$.

When $W \equiv 0$, we say simply that V is a Liapunov function for F . ■

Theorem 4.

Let F be a set-valued map from a closed subset $K \subset X$ to X , W be a non-negative function from graph (F) to $\mathbb{R}$ and V be a lower semicontinuous function from K to $\mathbb{R}_+$. Assume that V is a Liapunov function for F with respect to W . Then,

(6) $\forall\, \varepsilon > 0\ ,\ \exists\, x_\varepsilon \in K$ and $v_\varepsilon \in F(x_\varepsilon)$ satisfying $W(x_\varepsilon, v_\varepsilon) \leq \varepsilon\|v_\varepsilon\|$.

If we assume moreover that V is lower semicompact (this means that for all $\lambda \in \mathbb{R}$, the subsets $\{x \in K \mid V(x) \leq \lambda\}$ are relatively compact), then

(7) $\exists\, x_* \in K$, $\exists\, v_* \in F(x_*)$ such that $W(x_*,v_*) = 0$. ■

Proof.

a). We apply Ekeland's variational principle: $\forall\, \varepsilon > 0$, $\exists\, x_\varepsilon \in K$ such that, $\forall\, v \in X$, $0 \leq D_+V(x_\varepsilon)(v) + \varepsilon\|v\|^2$. Since V is a Liapunov function with respect to W, there exists $v_\varepsilon \in F(x_\varepsilon)$ such that $D_+V(x_\varepsilon)(v_\varepsilon) \leq -W(x_\varepsilon,v_\varepsilon)$. Hence property (6) holds true.

b). When V is also lower semicompact, there exists $x_* \in K$ that achieves the minimum of V . So $\forall\, v \in X$, $0 \leq D_+V(x_*)(v)$ by the variational principle. Since V is a Liapunov function, we choose $v_* \in F(x_*)$ such that $D_+V(x_*)(v_*) \leq -W(x_*,v_*)$. ■

So, Theorem 1 is the particular case when $W(x,v) \doteq \|v\|$. By taking $W(x,v) \doteq \|v\|^\alpha$, $\alpha > 1$, we obtain the existence of approximate stationary points:

Corollary 1

Let F be a set-valued map from a closed subset $K \subset X$ to X and V be a lower semicontinuous Liapunov function for F with respect to $\|\cdot\|^\alpha$, $\alpha > 1$. Then,

(8) $\forall\, \varepsilon > 0$, $\exists\, x_\varepsilon \in K$ such that $F(x_\varepsilon) \cap \varepsilon B \neq \emptyset$. ■

Note also that when W satisfies the condition

(9) $\forall\, x \in K$, $\forall\, v \neq 0$, $W(x,v) > 0$

we obtain the existence of stationary points.

Corollary 2

Let F be a set-valued map from a closed subset $K \subset X$ to X , $W:\mathrm{graph}(F) \to \mathbb{R}_+$ be a nonnegative function satisfying property (9) and V be a lower semicontinuous and lower semicompact liapunov function for F with respect to W . Then there exists a stationary point $x_* \in K$ of F . ■

Remarks

We shall prove that in this case, under some supplementary continuity assumptions, solutions to the differential inclusion $x' \in F(x)$, $x(0) = x_0$, converge to a stationary point of F in some sense when $t \to \infty$. Note that the tangential condition

(10) $\forall\, x \in K$, $F(x) \cap D_K(x) \neq \emptyset$

are involved in the Liapunov condition (5), since the domain of the upper contingent derivative $D_+V(x)(\cdot)$ is contained in the contingent cone $D_K(x)$ to the domain K of V : Indeed, if V is a Liapunov function, there exist $v \in F(x)$ such that

$D_+V(x)(v) \leq -W(x,v) \leq 0$. Hence $v \in \mathrm{Dom}\, D_+V(x)(\cdot) \subset D_K(x)$. Therefore the tangential condition (10) holds true.

Actually, if we take $V \doteq \phi_K$ the indicator of K , defined by $\phi_K(x) \doteq 0$ when $x \in K$ and by $\phi_K(x) \doteq \infty$ when $x \notin K$, the Liapunov condition can be written

(11) $\quad \forall\, x \in K \;,\quad \exists\, v \in F(x) \cap D_K(x)$ satisfying $W(x,v) = 0$.

We mention also that when K is convex and compact and when F is upper hemicontinuous with closed convex values, the Browder - Ky Fan theorem states that the tangential condition (10) is sufficient for establishing the existence of a stationary point $x_\star \in K$ of F (See for instance Aubin [3], Chapter 15). ■

Example

By taking for $V(x)$ the restriction to K of $x \to \frac{1}{2}\|x-x_0\|^2$, we obtain the following corollary, after noticing that

$$D_+ V(x)(v) = \langle x-x_0, v \rangle \quad \text{when} \quad v \in D_K(x) .$$

Corollary 3

Let K be a weakly closed subset of X and W be a function from X to $[0,\infty]$ that is strictly positive when $v \neq 0$. Let a set-valued map F from K to X and $x_0 \in X$ satisfy

(12) $\quad \forall\; x \in K \;,\quad \exists\, v \in F(x) \cap D_K(x)$ such that $\langle x-x_0, v \rangle - W(v) \leq 0$.

Then the best approximations of x_0 by elements of K are stationary points of F . ■

As a particular case, we obtain the following result. Let F^{-1} denote the inverse of F and $\sigma(F^{-1}(v),p) \doteq \sup\{\langle p,x \rangle \mid x \in F^{-1}(v)\}$ denote the support function of $F^{-1}(v)$.

Corollary 4

Let K be a weakly closed subset of X and $F:K \to X$ be a set-valued map satisfying

(13) $\quad \forall\, v \in F(K),\; v \neq 0$, then $\sigma(F^{-1}(v),v) < \langle x_0, v \rangle$

and

(14) $\quad \forall\, x \in K \;,\; F(x) \cap D_K(x) \neq \emptyset$.

Then the best approximations of x_0 by elements of K are stationary points of F .

Proof.

We apply Corollary 3 when W is defined by

$$(15)\qquad \begin{cases} W(v) = \langle x_0, v\rangle - \sigma(F^{-1}(v),v) & \text{when } v \in F(K),\ v \neq 0 \\ W(0) = 0 & \\ W(v) = +\infty & \text{when } v \notin F(K) \text{ and } v \neq 0 . \end{cases}$$

Then for any $v \in F(x)$, we deduce that $\langle x,v\rangle \leq \sigma(F^{-1}(v),v)$ since $x \in F^{-1}(v)$.

Therefore

$$\langle x - x_0, v\rangle \leq \sigma(F^{-1}(v),v) - \langle x_0, v\rangle \leq -W(v) . \qquad \blacksquare$$

12. Monotone trajectories of a differential inclusion

Let K and L be subsets of $X = \mathbb{R}^n$. Let $V: K \to \mathbb{R}_+$ and $W: K \times L \to \mathbb{R}$ be two given functions and $F: K \to X$ be a set-valued map.

Definition 1

We say that a trajectory $x(\cdot)$ of the differential inclusion

(1) $$x'(t) \in F(x(t)) \; ; \; x(0) = x_0$$

is monotone (with respect to V and W) if

(2) $$\forall s > t \geq 0 \,, \; V(x(s)) - V(x(t)) + \int_t^s W(x(\tau), x'(\tau))d\tau \leq 0 \,. \qquad \blacksquare$$

Note that this condition implicitely implies that

(3) $$\forall t \geq 0 \,, \; x(t) \in K \quad (\text{i.e.,} \quad x(\cdot) \text{ is "viable"})$$

since V is defined on K.

Proposition 1

If $W: K \times L \to \mathbb{R}_+$ is nonnegative, then

(4) $t \to V(x(t))$ decreases and converges to $\alpha \doteq \lim_{t \to \infty} V(x(t))$

and

(5) $$\int_0^\infty W(x(\tau), x'(\tau))d\tau \doteq \lim_{t \to \infty} \int_0^t W(x(\tau), x'(\tau))d\tau < +\infty \,. \qquad \blacksquare$$

Remark

Note that (4) implies that when V is lower semicontinuous, all the cluster points x_* of the trajectory when $t \to \infty$ (if any) satisfy $\alpha = V(x_*)$. $\blacksquare$

Proof.

The first statement is obvious.

Also, since

$$\int_t^s W(x(\tau), x'(\tau))d\tau \leq V(x(t)) - V(x(0)) \to \alpha - \alpha = 0$$

when $t,s \to \infty$, the Cauchy criterion implies that when W is nonnegative

$$\int_0^\infty W(x(\tau), x'(\tau))d\tau \doteq \lim_{t \to \infty} \int_0^t W(x(\tau), x'(\tau))d\tau < +\infty$$

(where the integral is a Riemann improper integral). $\blacksquare$

We shall see in Section 14 that this latter condition implies that $W(x(t), x'(t))$ converges to 0 in some sense when $t \to \infty$.

According to the assumptions relating V and W, monotonicity property (2) yields useful informations on the behavior of the trajectory.

Example Trajectories with finite length. Let us consider the case where

(6) $$W(x,u) \doteq \|v\|.$$

Theorem 1

The trajectories $x(\cdot)$ on $[0,\infty[$ that are monotone with respect to V and $W:(x,v) \to \|v\|$ have finite length $\int_0^\infty \|x'(\tau)\| d\tau$ and converge to $x_* \in \bar{K}$ when $t \to \infty$. If K is closed and F is upper semicontinuous with compact convex values, then x_* is a stationary point of F. ∎

Proof.

By (5), $\int_0^\infty \|x'(\tau)\| d\tau$, which is the length of the trajectory, is finite. Furthermore, inequality

$$\|x(t) - x(s)\| \leq \int_t^s \|x'(\tau)\| d\tau \to 0 \text{ when } t,s \to \infty$$

and the Cauchy criterion imply that $\lim_{t\to\infty} x(t) = x_*$ does exist. The following theorem shows that x_* is a stationary point.

Theorem 2.

Let F be an upper semicontinuous map from a closed subset $K \subset X$ to X with compact convex values and $x(\cdot)$ be a trajectory of the differential inclusion (1) that converges to some $x_* \in K$. Then x_* is a stationary point of F. ∎

Proof

Assume that $0 \notin F(x_*)$: there exists $\varepsilon > 0$ such that

(7) $$\varepsilon B \cap (F(x_*) + \varepsilon B) = \emptyset$$

(for $F(x_*)$ is a closed subset). Since F is upper semicontinuous, there exists $\delta > 0$ such that $F(y) \subset F(x_*) + \varepsilon B$ whenever $\|y - x_*\| \leq \delta$. Hence there exists $T > 0$ such that, $\forall\, t \geq T$, $\|x(t) - x_*\| \leq \delta$. Consequently:

(8) $$\forall\, t \geq T,\ F(x(t)) \subset F(x_*) + \varepsilon B.$$

Since $x'(t) \in F(x(t))$ for almost all $t \geq 0$, the mean value theorem implies that for all $t \geq T$

$$(9) \qquad \frac{x(t) - x(T)}{t - T} = \frac{1}{t-T}\int_T^t x'(\tau)d\tau \subset \overline{co}(F(x_*) + \varepsilon B) = F(x_*) + \varepsilon B$$

since $F(x_*) + \varepsilon B$ is closed and convex. Therefore, statements (7) and (9) imply that $\forall t \geq T, \left\| \frac{x(t) - x(T)}{t - T} \right\| \geq \varepsilon$, which is a contradiction of the fact that $x_* = \lim_{t \to \infty} x(t)$. ■

<u>Example.</u>

We shall illustrate the importance of monotone trajectories by the following theorem.

<u>Theorem 3</u>

Let φ be a continuous bounded function from $[0,\infty[$ to $\mathbb{R}$ and let $\dot{W}(x,v) = \varphi(V(x))$. Let x be a trajectory satisfying property (2) and $w(\cdot)$ be a solution to the differential equation

$$(10) \qquad w'(t) + \varphi(w(t)) = 0 \quad \forall t \geq 0 \ , \quad w(0) = V(x(0)) \ .$$

Then, we obtain the following estimate:

$$(11) \qquad \forall t \geq 0 \ , \quad V(x(t)) \leq w(t).$$ ■

This statement is an obvious consequence of the following Theorem, of which we give a proof due to H. Antosiewicz.

<u>Theorem 4</u>

Let $\Omega \subset \mathbb{R}$ be an open interval and $T \leq +\infty$. We consider a function φ from $[0,T[\times \Omega$ to $\mathbb{R}$ satisfying the following properties.

$$(12) \qquad \begin{cases} \text{i)} & \forall t \in [0,T[\ , \ x \mapsto \varphi(t,x) \ \text{is continuous} \\ \text{ii)} & \forall x \in \Omega \ , \ t \mapsto \varphi(t,x) \ \text{is measurable} \\ \text{iii)} & \exists a \in L^1(0,T) \ \text{such that,} \ \forall (t,x) \in [0,T[\times \Omega, \ |\varphi(t,x)| \leq a(t). \end{cases}$$

Let $v : [0,T[\to \Omega$ be a continuous function satisfying

$$(13) \qquad \forall s > t \ , \quad v(s) - v(t) + \int_t^s \varphi(\tau,v(\tau))d\tau \leq 0 \ .$$

Then, there exists a maximal interval $[0,T_1[$ such that the differential equation $w' + \varphi(w) = 0$, $w(0) = v(0)$ has at least one solution w on $[0,T_1]$ satisfying

$$(14) \qquad \forall t \geq 0 \ , \quad v(t) \leq w(t).$$ ■

Proof.

a). We introduce the subsets

$$K = \{(t,x) \in [0,T[\times \Omega \quad \text{such that} \quad x \geq v(t)\}$$

$$L = \{(t,x) \in [0,T[\times \Omega \quad \text{such that} \quad x \leq v(t)\} .$$

Since $K \cup L = [0,T[\times \Omega$ and $K \cap L = \{(t,x) \mid x = v(t)\}$, we can define a function Ψ on $[0,T[\times \Omega$ by

$$\text{(15)} \qquad \Psi(t,x) = \begin{cases} \varphi(t,x) & \text{if} \quad x \geq v(t) \quad (\text{i.e., if } (t,x) \in K) \\ \varphi(t,v(t)) & \text{if} \quad x \leq v(t) \quad (\text{i.e., if } (t,x) \in L). \end{cases}$$

b). The function Ψ inherits the properties of φ. To see this, we associate with any $t \in [0,T[$ the subsets

$$K(t) \doteq \{x \in \Omega \mid x \geq v(t)\}, \quad L(t) \doteq \{x \in \Omega \mid x \leq v(t)\}.$$

They are closed and cover Ω. Thus, when $K(t) \neq \emptyset$ (resp. $L(t) \neq \emptyset$), the restriction of $\Psi(t,\cdot)$ to $K(t)$ (resp., to $L(t)$) is continuous. Hence we conclude that for all $t \in [0,T[$, $x \mapsto \Psi(t,x)$ is continuous. Similarly, we introduce the subsets:

$$K(x) \doteq \{t \in [0,T[\mid x \geq v(t)\} \quad \text{and} \quad L(x) \doteq \{t \in [0,T[\mid x \leq v(t)\} .$$

Hence, when $K(x) \neq \emptyset$ (resp. $L(x) \neq \emptyset$), the restriction of $\Psi(\cdot,x)$ to $K(x)$ (resp. $L(x)$) is measurable. Consequently, for all $x \in \Omega$, the function $t \to \Psi(t,x)$ is measurable. Obviously, $|\Psi(t,x)| \leq a(t)$ for all $(t,x) \in [0,T[\times \Omega$.

c). We choose now $T_0 < T$ such that the set of points $(t,x) \in [0,T_0] \times \mathbb{R}$ satisfying $|x-x_0| \leq \int_0^t a(\tau)d\tau$ is contained in $[0,T[\times \Omega$. Then, by Caratheodory's theorem, there exists at least a solution $w(\cdot):[0,T_0] \to \Omega$ to the differential equation

$$\text{(16)} \qquad w'(t) + \Psi(t,w(t)) = 0 \; ; \quad w(0) = v(0).$$

d). We assert that for all $t \in [0,T_0]$, $v(t) \leq w(t)$. If not, there would exist $t_2 \in]0,T_0]$ such that $v(t_2) > w(t_2)$. Let $t_1 \doteq \inf\{s \in [0,T_0] \mid v(t) > w(t)$ for all $t \in [s,t_2]\}$. By continuity, we have $w(t_1) = v(t_1)$ and $v(t) > w(t)$ for all $t \in]t_1,t_2]$. Since w is a solution to (16), then

$$w(t_2) = w(t_1) - \int_{t_1}^{t_2} \Psi(s,w(s))ds .$$

Since $v(s) > w(s)$, we deduce that $\psi(s,w(s)) = \varphi(s,v(s))$ for all $s \in]t_1,t_2]$.
Hence

$$w(t_2) = v(t_1) - \int_{t_1}^{t_2} \varphi(s,v(s))ds \ .$$

By assumption (13), we deduce that $w(t_2) \geq v(t_2)$ (because $t_1 \leq t_2$). This contradicts inequality $v(t_2) > w(t_2)$. Hence $v(t) \leq w(t)$ on $[0,T_0]$. ∎

13. Almost convergence of monotone trajectories to stationary points.

We recall that when a Liapunov function V for F (with respect to W) satisfies

(1) V is lower semicontinuous and lower semicompact,

there exists x_* and v_* satisfying

(2) $$x_* \in K\ ,\quad v_* \in F(x_*) \quad \text{and} \quad W(x_*,v_*) = 0\ .$$

So, when the function W satisfies the property

(3) $$\forall\, x \in K\ ,\quad \forall\, v \neq 0\ ,\quad W(x,v) > 0$$

the existence of a Liapunov function V for F with respect to W and assumptions (1) and (3) imply the existence of stationary points.

We wish to prove that cluster points x_* and v_* of $x(t)$ and $x'(t)$ when $t \to \infty$ do satisfy the property (2) [In this case, property (3) guarantees that such cluster points x_* of $x(t)$ are stationary points of F].

We already noted that when $W(x,v) = \|v\|$, the trajectory $x(t)$ converges to a limit x_*, which is a stationary point by Theorem 11.2. The proof of this theorem does not yield the fact that cluster points of $x(t)$ are stationary points.

A difficulty arises right now: the derivative $x'(t)$ is only defined almost everywhere. So we shall use an adaptation proposed by A. Cellina of the concepts of limit and cluster points for measurable functions to prove that measurable functions taking their values in a compact subset do have such "almost" cluster points and that "almost cluster points" x_* and v_* of $x(t)$ and $x'(t)$ satisfy $W(x_*,v_*)$ when $x(\cdot)$ is a monotone trajectory with respect to V and W.

Let $\mu(A)$ denote the Lebesgue measure of a subset $A \subset \mathbb{R}$.

Definition 1

Let $x : [0,\infty[\to X$ be a measurable function and $x_* \in X$. We say that x_* is the almost limit of $x(\cdot)$ when $t \to \infty$ (and we write $x_* = \underset{t\to\infty}{\text{a lim}}\, x(t)$) if

(4) $$\forall\, \varepsilon > 0\ ,\quad \exists T > 0 \text{ such that } \mu\{t \in [T,\infty[\ \mid\ \|x(t) - x_*\| \geq \varepsilon\} = 0\ .$$

We say that x_* is an almost cluster point of $x(t)$ when $t \to \infty$ if

(5) $$\forall\, \varepsilon > 0\ ,\quad \mu\{t \in [0,\infty[\ \mid \|x(t) - x_*\| \leq \varepsilon\} = \infty\ .$$ ■

These concepts are justified by the following theorem:

Theorem 1

Let F be an upper semicontinuous map from $K \subset \mathbb{R}^n$ to the compact subsets of $\mathbb{R}^n$, W be a nonnegative lower semicontinuous function defined on graph (F) and V be a nonnegative lower semicontinuous and lower semicompact function defined on K. For any monotone trajectory $x(t)$ of F with respect to V and W, the functions $x(t)$ and $x'(t)$ have almost cluster points x_* and v_* which satisfy

(6) $\qquad x_* \in K\ ,\quad v_* \in F(x_*)$ and $W(x_*,v_*) = 0$.

If W satisfies the condition

(3) $\qquad \forall\, x \in K\ ,\quad \forall\, v \neq 0\ ,\quad W(x,v) > 0$

then such an almost cluster point x_* is a stationary point. ■

The proof of this theorem will be obtained by tying up the following properties of almost convergence. For simplicity, we restrict our study to the case of functions of a real variable. Adaptation for functions defined on a measured space is quite easy.

We begin by showing that the usual concepts of limit and cluster point are particular cases of almost limit and almost cluster point.

Proposition 1.

Any limit of x_* of $x(\cdot):[0,\infty[\ \to X$ is an almost limit point. If $x(\cdot)$ is uniformly continuous, any cluster point x_* of $x(\cdot)$ is an almost cluster point. ■

Proof.

a) To say that $x_* = \lim\limits_{t \to \infty} x(t)$ amounts to saying that $\forall\, \varepsilon > 0\ ,\ \exists\ T > 0$ such that $[T,\infty[\ \cap \{t \mid \|x(t) - x_*\| \geq \varepsilon\} = \emptyset$. Hence the measure of this set is equal to 0 .

b) Let x_* be a cluster point of $x(\cdot)$: Since $x(\cdot)$ is uniformly continuous, there exists η such that $|s-t| \leq \eta$ implies $\|x(t) - x(s)\| \leq \varepsilon/2$. Also, there exists a sequence $t_n \to \infty$ (which satisfies $t_{n+1} - t_n \geq 2\eta$) such that $\|x_*-x(t_n)\| \leq \varepsilon/2$ when $n \geq N_\varepsilon$. So, for any $n \geq N_\varepsilon$, the disjoint intervals $[t_n - \eta,\ t_n + \eta\,]$ are contained in $\{t \in [0,\infty[\,|\,\|x(t) - x_*\| \leq \varepsilon\}$. Hence

$$\infty = \mu\Big(\bigcup_{n \geq N_\varepsilon} [t_n - \eta, t_n + \eta]\Big) \leq \mu(\{t \in [\cdot,\infty[\,|\,\|x(t) - x_*\| \leq \varepsilon\} \ . \qquad ■$$

The following example justifies the introduction of the concept of almost cluster point.

Proposition 2.

If W is a nonnegative function belonging to $L^1(0,\infty)$, then 0 is an almost cluster point of W when $t \to \infty$. ∎

Proof.

If not, there exists $\varepsilon > 0$ such that the measure of $A_\varepsilon \doteq \{t \in [0,\infty[\mid W(t) \leq \varepsilon\}$ is finite. Hence the measure of $\complement A_\varepsilon = \{t \in [0,\infty[\mid W(t) > \varepsilon\}$ is infinite. Therefore:

$$\int_0^\infty w(\tau)d\tau \geq \int_{A_\varepsilon} w(\tau)d\tau + \int_{\complement A_\varepsilon} w(\tau)d\tau \geq \varepsilon\, \mu(\complement A_\varepsilon) = \infty$$

which is a contradiction. ∎

Proposition 3

An almost limit x_* of a measurable function $x(\cdot):[0,\infty[\to X$ is the unique almost cluster point. ∎

Proof.

Let y_* be an almost cluster point different from x_*. We choose $\varepsilon < \|x_* - y_*\|/2$ and T such that the subset $K \doteq \{t \in [T,\infty[\mid \|x(t) - x_*\| \geq \varepsilon\}$ has a measure equal to 0.

The subset $L = \{t \in [T,\infty[\mid \|x(t) - y_*\| \leq \varepsilon\}$ is obviously contained in K and has an infinite measure for y_* is an almost cluster point. Hence $\infty = \mu(L) \leq \mu(K) = 0$, which is impossible. So, $x_* = y_*$. ∎

Proposition 4

Let f be a continuous (single valued) map from X to Y and $x(\cdot):[0,\infty[\to X$ be a measurable function. If x_* is the almost limit (resp. an almost cluster point) of $x(\cdot)$, then $f(x_*)$ is the almost limit (resp. an almost cluster point) of $f(x(\cdot))$. ∎

Proof. *It is left as an exercise.* ∎

Theorem 2 (Cellina)

Let K be a compact subspace of X and $x(\cdot):[0,\infty[\to K$ be a measurable function. There exists an almost cluster point $x_* \in K$ of $x(\cdot)$ when $t \to \infty$. ∎

Proof.

We define inductively decreasing sequences of measurable sets $\Delta_n \subset [0,\infty[$ and of closed subsets $E_n \subset K$ such that

(7) $$\Delta_n = x^{-1}(E_n), \quad \mu(\Delta_n) = \infty, \quad \operatorname{diam}(E_n) \leq 1/n .$$

For $n = 1$, we cover the compact set K with a finite number of sets B_j^1 of diameter at most 1. Thus the subsets $x^{-1}(B_j^1)$ cover $[0,\infty[$ and, consequently, one of them, denoted Δ_1, has an infinite measure. We set E_1 the corresponding set B_j^1.

Having defined the subsets Δ_k and E_k up to n, we cover the compact set E_n by a finite number of closed subsets B_j^{n+1} of diameter at most $1/n+1$.

Their preimages $x^{-1}(B_j^n)$ form a finite covering of Δ_n. Since $\mu(\Delta_n) = \infty$, at least one of these sets, denoted Δ_{n+1}, has an infinite measure. Call E_{n+1} the corresponding B_j^{n+1}. Hence $\bigcap_{n\geq 0} E_n = \{x_*\}$. It remains to show that x_* is an almost cluster point. Fix $\varepsilon > 0$ and $T > 0$. Then, a neighborhood $N_\varepsilon(x_*)$ contains the subsets E_n for $n \geq n(\varepsilon)$. Consequently, $x^{-1}(N_\varepsilon(x_*)) \supset x^{-1}(E_n) = \Delta_n$ for all $n \geq n_0(\varepsilon)$. Hence,

$$\mu\{t \in [0,\infty[\,|\, x(t) \in N_\varepsilon(x_*)\} \geq \mu_1(\Delta_n) = \infty. \qquad \blacksquare$$

Proposition 5

Let W be a nonnegative lower semicontinuous from $L \subset X$ to $\mathbb{R}$. If $x(\cdot)$ is a measurable function from $[0,\infty[$ to L such that

(8) $$\int_0^\infty W(x(\tau))d\tau < +\infty,$$

then any almost cluster point x_* of $x(t)$ when $t \to \infty$ satisfies the equation $W(x_*) = 0$. $\blacksquare$

Proof.

Let x_* be an almost cluster point of $x(\cdot)$ when $t \to \infty$ and assume that $W(x_*) > 0$. We take $\varepsilon = W(x_*)/2 > 0$. Since W is lower semicontinuous, there exists η such that $W(x_*)/2 \leq W(y)$ when $\|y - x_*\| \leq \eta$. So the subset $A_\eta = \{t \in [0,\infty[\mid \|x(t) - x_*\| \leq \eta\}$, whose measure is infinite, is contained in the set $B_\varepsilon = \{t \in [0,\infty[\,|\,W(x_*)/2 \leq W(x(t))\}$. Hence

$$\int_0^\infty W(x(\tau))d\tau \geq \int_{B_\varepsilon} W(x(\tau))d\tau \geq \frac{W(x_*)}{2}\mu(B_\varepsilon) = \infty.$$

This is a contradiction. ∎

We are ready to prove Theorem 1.

<u>Proof of Theorem</u> 1

Since V is lower semicontinuous and lower semicompact, then $x(t)$ remains in the compact subset $Q \doteq \{x \in K | V(x) \leq V(x_0)\}$. Because F is upper continuous with compact values, the set

$$F_Q = \{(x,v) \in \mathbb{R}^n \times \mathbb{R}^n \mid x \in Q, \quad v \in F(x)\}$$

which is the graph of the restriction $F|_Q$ of F to Q, is compact. Hence the function $t \to (x(t),x'(t))$ is a measurable function taking its values in the compact set F_Q.

By <u>Theorem</u> 2, there exists an almost cluster point $(x_*,v_*) \in F_Q$. Since $x(\cdot)$ is a monotone trajectory with respect to V and W, we know that $\int_0^\infty W(x(\tau),x'(\tau))d\tau < +\infty$.

Hence <u>Proposition</u> 5 implies that $W(x_*,v_*) = 0$. ∎

14. <u>Necessary condition for the existence of monotone trajectories</u>

We shall prove that the existence of monotone trajectories with respect to V and W of the differential inclusion $x' \in F(x)$, $x(0) = x_0$ for all initial value $x_0 \in K$ implies that V is a Liapunov function for F with respect to W.

<u>Theorem 1</u>

Let F be an upper semicontinuous map from $K \subset \mathbb{R}^n$ to $\mathbb{R}^n$ with compact convex values. Let W be a lower semicontinuous nonnegative function on $K \times \text{co}\, F(K)$ which is convex with respect to v. We assume that for all $x_0 \in K$, there exist $T > 0$ and a trajectory $x(\cdot)$ on $[0,T[$ of the differential inclusion $x' \in F(x)$, $x(0) = x_0$ satisfying

(1) $$\forall s > t, \quad V(x(s)) - V(x(t)) + \int_t^s W(x(\tau),x'(\tau))d\tau \leq 0 .$$

Then, V is a Liapunov function for F with respect to W:

(2) $$\forall x \in K, \quad \exists v \in F(x) \text{ such that } D_+V(x)(v) + W(x,v) \leq 0 . \qquad \blacksquare$$

<u>Proof.</u>

It is analogous to the proof of Haddad's Proposition. Since F is upper semicontinuous, we can associate to $\varepsilon > 0$ an $\eta > 0$ such that, for all $\tau \leq \eta$, $F(x(\tau)) \subset F(x_0) + \varepsilon B$, which is compact and convex. Since $\frac{x(h) - x_0}{h} = \frac{1}{h}\int_0^h x'(\tau)d\tau$ and since, for almost all τ, $x'(\tau)$ belongs to $F(x_0) + \varepsilon B$, the mean-value theorem implies that $\frac{x(h) - x_0}{h}$ belongs to this compact subset.

Hence there exists a subsequence $h_n \to 0$ such that $v_n \doteq \frac{x(h_n) - x_0}{h_n}$ converges to some v_0 in $F(x_0) + \varepsilon B$.

Since this inclusion holds true for all $\varepsilon > 0$, we deduce that $v_0 \in F(x_0)$.

We also observe that $x_0 + h_n v_n = x(h_n)$ belongs to K, the domain of V.

Hence property (1) implies that

(3) $$\frac{V(x_0 + h_n v_n) - V(x_0)}{h_n} + \frac{1}{h_n}\int_0^{h_n} W(x(\tau),x'(\tau))d\tau \leq 0 .$$

Let us assume for a while that the following properties hold true:

<u>Proposition 1</u>

Let W be a lower semicontinuous function on $K \times L$ which is convex with respect to $v \in L$. Let $x \in C(0,T;K)$ and $v \in L^\infty(0,T,M)$ be given, where M is compact. Let

(4) $$x_0 \doteq \lim_{t\to 0} x(t) \quad \text{and} \quad v_0 \doteq \lim_{h_n \to 0} \frac{1}{h_n} \int_0^{h_n} v(t)dt .$$

Then,

(5) $$W(x_0,v_0) \leq \liminf_{h_n \to 0} \frac{1}{h_n}\int_0^{h_n} W(x(\tau),v(\tau))d\tau. \quad \blacksquare$$

<u>End of the proof of Theorem 1.</u>

By <u>Proposition 1</u>, $W(x_0,v_0) \leq \liminf\limits_{h_n \to 0} \frac{1}{h_n} \int_0^{h_n} W(x(\tau),x'(\tau))d\tau$. Hence

$$D_+ V(x_0)(v_0) + W(x_0,v_0) \leq \liminf_{\substack{h_n \to 0+ \\ v_n \to v_0}} \frac{V(x_0 + h_n v_n) - V(x_0)}{h_n} + \liminf_{h_n \to 0+} \frac{1}{h_n} \int_0^{h_n} W(x(\tau),x'(\tau))d\tau \leq$$

$$\leq \liminf_{\substack{h_n \to 0+ \\ v_n \to v_0}} \left[\frac{V(x_0 + h_n v_n) - V(x_0)}{h_n} + \frac{1}{h_n} \int_0^{h_n} W(x(\tau),x'(\tau))d\tau\right] \leq 0 . \quad \blacksquare$$

Before proving Proposition 1, we need the following result of Ekeland - Temam [1].

When M is a subset of X, we denote by $W_M(x,\cdot)$ the restriction to M of the function $v \mapsto W(x,v)$ and by $\mathcal{E}p W_M(x,\cdot)$ its epigraph.

<u>*Proposition* 2.</u>

Let K and L be two nonempty subsets of X and $W:K \times L \to \mathbb{R}$ satisfy

(5) $$\begin{cases} \text{i)} & W \text{ is lower semicontinuous} \\ \text{ii)} & \forall x \in K ,\ v \mapsto W(x,v) \text{ is convex .} \end{cases}$$

Then, for any compact convex subset $M \subset L$, the restriction $W_M(x,\cdot)$ of $v \to W(x,v)$ to M satisfies the property

(6) $x \to \mathcal{E}p W_M(x,\cdot)$ is upper semicontinuous with closed convex values. $\blacksquare$

<u>Proof of Proposition</u> 2

To say that $x \to \mathcal{E}p\, W_M(x,\cdot)$ is upper semicontinuous at x_0 means that for all $\varepsilon > 0$, there exists $\eta > 0$ such that, $\forall x \in K \cap (x_0 + \eta B)$, $\forall v \in M$, $\exists v_0 \in M \cap (v + \varepsilon B)$ satisfying $W(x_0,v_0) \leq W(x,v) + \varepsilon$. Since W is lower semicontinuous, we can associate to each $v \in M$ an $\eta_v \in]0,\varepsilon[$ such that

(7) $$\forall x \in x_0 + \eta_v B,\ \forall w \in v + \eta_v B,\ W(x_0,v) \leq W(x,w) + \varepsilon .$$

Since M is compact, it is covered by a finite number p of balls $v_i + \eta_{v_i} B$.

Let $\eta \doteq \min_{i=1,\dots,p} \eta_{v_i}$. Hence for all $x \in x_0 + \eta B$, for all $v \in M$, there exists v_i such that $\|v - v_i\| \le \eta_{v_i} \le \varepsilon$ satisfying $W(x_0, v_i) \le W(x,v) + \varepsilon$. Since $v \to W_M(x,v)$ is convex and continuous, then $\mathcal{E}p\, W(x,\cdot)$ is closed and convex. ■

We are now ready to prove Proposition 1.

<u>Proof of Proposition</u> 1

By <u>Proposition</u> 2, $x \to \mathcal{E}p\, W_M(x,\cdot)$ is upper semicontinuous. Then there exists h_ε such that, for all $t \in [0,h_\varepsilon]$, $\mathcal{E}p W(x(t),\cdot) \subset \mathcal{E}p W_M(x_0,\cdot) + \varepsilon(B\times B)$. Therefore, for all $\tau \in [0,h_\varepsilon]$, $(v(\tau), W(x(\tau),v(\tau))) \in \mathcal{E}p\, W_M(x(\tau),\cdot) \subset \mathcal{E}p\, W_M(x_0,\cdot) + \varepsilon(B\times B)$. Hence, by the mean-value theorem, we deduce that, for all $h_n \le h_\varepsilon$,

$$(8) \qquad \left(\frac{1}{h_n}\int_0^{h_n} v(\tau)d\tau\ ,\ \frac{1}{h_n}\int_0^{h_n} W(x(\tau),\ v(\tau))d\tau\right) \in \overline{co}(\mathcal{E}p\, W_M(x_0,\cdot) + \varepsilon(B\times B))\ .$$

Therefore, by taking the limit, we obtain:

$$(9) \qquad \left(v_0,\ \liminf_{h_n \to 0} \frac{1}{h_n}\int_0^{h_n} W(x(\tau),v(\tau))d\tau\right) \in \overline{co}(\mathcal{E}p\, W_M(x_0,\cdot) + \varepsilon(B\times B)).$$

Since this is true for all $\varepsilon > 0$ and since $\mathcal{E}p\, W_M(x_0,\cdot)$ is closed and convex, we get:

$$(10) \qquad W(x_0,v_0) \le \liminf_{h_n \to 0} \frac{1}{h_n}\int_0^{h_n} W(x(\tau),v(\tau))d\tau. \qquad ■$$

15. Sufficient conditions for the existence of monotone trajectories.

We shall prove that, conversely, if V is a Liapunov function for F with respect to W there exist monotone trajectories of the differential inclusion $x' \in F(x)$ that are monotone with respect to V and W.

Theorem 1

Let K be a closed subset of $\mathbb{R}^n$ and F be an upper set-valued map from K to the nonempty compact convex subsets of $\mathbb{R}^n$. Let W be a function defined on $K \times \text{co}\, F(K)$ satisfying

$$(1) \quad \begin{cases} \text{i)} & W \text{ is nonnegative and continuous} \\ \text{ii)} & \forall\, x \in K\,,\ v \to W(x,v) \text{ is convex.} \end{cases}$$

Let $V : K \to \mathbb{R}_+$ be a Liapunov function with respect to F and W:

$$(2) \qquad \forall\, x \in K\,,\quad \exists v \in F(x) \text{ such that } D_+ V(x)(v) + W(x,v) \le 0\ .$$

We also assume that

(3) $\quad V$ is continuous (for the topology induced on K).

Then, for every $x_0 \in K$, there exist $T > 0$ and a monotone trajectory $x(\cdot)$ on $[0,T[$ of the differential inclusion $x' \in F(x)$ and $x(0) = x_0$.

If $F(K)$ is bounded, then we can take $T = \infty$. ■

Proof.

It is analogous to the proof of Haddad's theorem (See Haddad [1]). Since K is locally compact, there exist $r > 0$ such that $K_0 = K \cap (x_0 + rB)$ is compact. We set $T = r/(\|F(K_0)\| + 1)$. If $F(K)$ is bounded, we take T arbitrary and we set $K_0 = K \cap c\ell(x_0 + T\,F(K) + B)$, which is compact.

Let us take $y \in K$ and $v_y \in F(y)$ satisfying

$$(4) \qquad D_+ V(y)(v_y) + W(y,v_y) \le 0\ .$$

This is possible thanks to assumption (2).

By the very definition of $D_+ V(y)(v_y)$, we know that there exist $h_y \leq 1/k$ and $u_y \in v_y + 1/kB$ such that

$$(5) \qquad \frac{V(y + h_y u_y) - V(y)}{h_y} \leq D_+ V(y)(v_y) + 1/k.$$

Hence

$$(6) \qquad \frac{V(y + h_y u_y) - V(y)}{h_y} + W(y, v_y) < \frac{1}{k} .$$

We set

$$(7) \qquad N(y) = \left\{ x \in K \,\middle|\, \frac{V(x + h_y u_y) - V(x)}{h_y} + W(y, v_y) < \frac{1}{k} \right\}$$

Since the function V is continuous, the sets $N(y)$ are open and $y \in N(y)$. So we can find a ball of radius $\alpha_y \in]0, \frac{1}{k}]$ such that $(y + \alpha_y B) \cap K \subset N(y)$.

Since K_0 is compact, it can be covered by q such balls $y_j + \alpha_{y_j} B$. We set $\alpha_j \doteq \alpha_{y_j}$, $h_j \doteq h_{y_i}$, $u_j \doteq u_{y_j}$ and $v_j \doteq v_{y_j}$. Let us take now any $x \in K_0$. It belongs to a ball $y_j + \alpha_j B$ for some $j = 1, \ldots, q$. Then there exists u_j such that

$$(8) \qquad \begin{cases} \text{i)} & \dfrac{V(x + h_j u_j) - V(x)}{h_j} + W(y_j, v_j) < \dfrac{1}{k} \\ \\ \text{ii)} & \|x - y_j\| \leq \alpha_j \leq \dfrac{1}{k} \quad , \quad \|u_j - v_j\| \leq \dfrac{1}{k} . \end{cases}$$

Let $h_0(k) = \min\limits_{j=1,\ldots,q} h_j > 0$. So, cancelling the index j, we have proved that for all $x \in K_0$, there exist $h \in [h_0(k), \frac{1}{k}]$ and $u \in \mathbb{R}^n$ satisfying the two *following properties*

$$(9) \qquad \begin{cases} \text{i)} & \exists\, y \in K \text{ and } v \in F(y) \text{ such that } \|x - y\| \leq \dfrac{1}{k} , \|u - v\| \leq \dfrac{1}{k} \\ \text{ii)} & \dfrac{V(x + h\,u) - V(x)}{h} + W(y, v) \leq \dfrac{1}{k} . \end{cases}$$

Therefore, we can construct inductively a sequence of elements $x_p \in K_0$, $v_p \in F(y_p)$, $h_p \in [h_0(k), \frac{1}{k}]$ and $u_p \in \mathbb{R}^n$ satisfying

$$(10) \qquad \begin{cases} \text{i)} & x_{p+1} = x_p + h_p u_p \in K_0 \\ \text{ii)} & (x_p, u_p) \in (y_p, v_p) + \dfrac{1}{k}(B\times B) \subset \text{graph}(F) + \dfrac{1}{k}(B\times B) \\ \text{iii)} & \dfrac{V(x_{p+1}) - V(x_p)}{h_p} + W(y_p, v_p) < \dfrac{1}{k} . \end{cases}$$

We are sure that there exists an integer m such that

$$h_0 + h_1 + \dots + h_m \le T < h_1 + \dots + h_m + h_{m+1}$$

for $h_p \in [h_0(k), \frac{1}{k}]$ for all p.

Let us set $\tau_k^q = h_0 + \dots + h_q$. We interpolate this sequence by the piecewise linear function $x_k(\cdot)$ defined on each interval $]\tau_k^{p-1}, \tau_k^p[$ by $x_k(t) \doteq x_{p-1} + (t - \tau_k^{p-1})u_{p-1}$. We denote by $y_k(\cdot)$ and $v_k(\cdot)$ the step functions defined on this interval by $y_k(t) = y_p$ and $v_k(t) = v_p$.

When t is fixed in $]\tau_k^{p-1}, \tau_k^p[$, we have $|t - \tau_k^p| \le 1/k$ and there exists $(y_p, v_p) \in \text{graph}(F)$ such that $\|x_k'(t) - v_p\| = \|u_p - v_p\| \le 1/k$ and $\|x_k(t) - y_p\| \le \|x_k(t) - x_p\| + \|x_p - y_p\| \le |t - \tau_k^p| (\|u_p - v_p\| + \|v_p\|) + \|x_p - y_p\| \le 1/k\ (\|F(K_0)\| + 2)$.

By setting $\tilde{F}(t,x) \doteq F(x)$, we have proved that $\forall\, t \ge 0$,

$$(11) \qquad (t, x_k(t), x_k'(t)) \in (t, y_k(t), v_k(t)) + \varepsilon(k)(B \times B \times B) \subset \text{graph}(\tilde{F}) + \varepsilon(k)(B \times B \times B).$$

where $\varepsilon(k) \to 0$ when $k \to \infty$. We also know that $\|x_k'(t)\| \le \|F(K_0)\| + 1$ and $x_k(t) \in \overline{co}(K_0)$, which is compact. Hence the assumptions of the convergence theorem (See Aubin-Cellina-Nohel [1]) are satisfied: A subsequence of $x_k(\cdot)$ converges uniformly over compact intervals to a solution $x(\cdot)$ of the differential inclusion $x' \in F(x)$. Moreover, the sequence of derivatives $x_k'(\cdot)$ converges to $x'(\cdot)$ in $L^\infty(0,T;\mathbb{R}^n)$ supplied with the weak topology $\sigma(L^\infty, L^1)$.

On each point τ_k^p of the grid, the following inequality hold:

$$(12) \qquad V(x_k(\tau_k^{p+1})) - V(x_k(\tau_k^p)) + h^p W(x_k(\tau_k^p), v_k(\tau_k^p)) \le h^p/k.$$

By summing these inequalities from $p=q$ to $p=r-1$, we obtain,

$$V(x_k(\tau_k^r)) - V(x_k(\tau_k^q)) + \sum_{p=q}^{r-1} h^q W(x_k(\tau_k^p), v_k(\tau_k^p)) \le \frac{(\tau_k^r - \tau_k^q)}{k}.$$

We remark that $v_k(\tau) = x_k'(\tau_k^p)$ on the interval $[\tau_k^p, \tau_k^{p+1}[$. So, we can write the above inequality in the form:

$$(13) \qquad V(y_k(\tau_k^r)) - V(y_k(\tau_k^q)) + \int_{\tau_k^q}^{\tau_k^n} W(y_k(\tau), v_k(\tau))\,d\tau \le \frac{(\tau_k^r - \tau_k^q)}{h}.$$

We recall that $x_k(\cdot)$ converges to $x(\cdot)$ uniformly on compact intervals; so does $y_k(\cdot)$. We also know that $x_k'(\cdot)$ converges weakly to $x'(\cdot)$ in $L^\infty(0,T;X)$; so does $v_k(\cdot)$.

We assume for a while that the following Proposition holds true (See Ekeland-Temam [1]).

Proposition 1

Assume that the function $W : K \times L \to \mathbb{R}_+$ is nonnegative, lower semicontinuous and convex with respect to v. Then, for all compact convex subset $M \subset L$, the functional $\underset{\sim}{W}$ defined by

$$(14) \qquad \underset{\sim}{W}(x,v) \doteq \int_0^\infty W(x(\tau),v(\tau))d\tau \in [0,\infty]$$

is lower semicontinuous on $C(0,\infty;K) \times L^\infty(0,\infty;M)$ when $C(0,\infty;X)$ is supplied with the topology of uniform convergence on compact intervals and $L^\infty(0,\infty;M)$ with the topology induced by the weak topology $\sigma(L^\infty,L^1)$ on $L^\infty(0,\infty;X)$. ■

End of the proof of Theorem 1

By Proposition 1, we deduce that for all $s > t$,

$$(15) \qquad \int_t^s W(x(\tau),x'(\tau))d\tau \le \liminf_{k\to\infty} \int_t^s W(y_k(\tau),v_k(\tau))d\tau \ .$$

Since we can approximate s by τ_k^r and t by τ_k^q with $\tau_k^r \ge \tau_k^q$ for k large enough, we deduce from the continuity of V that $V(x_k(\tau_k^r))$ converges to $V(x(s))$ and $V(x_k(\tau_k^q))$ converges to $V(x(t))$. Also, since W is continuous, it is bounded on $K_0 \times F(K_0)$ and thus, $\int_{\tau_k^q}^{\tau_k^r} W(x(\tau),x'(\tau))d\tau$ converges to $\int_t^s W(x(\tau),x'(\tau))d\tau$. Hence, we can take the limit when $k \to \infty$ in inequalities (13): we find that $x(\cdot)$ satisfies the monotonicity condition

$$\forall\, s > t', \ V(x(s)) - V(x(t)) + \int_t^s W(x(\tau),x'(\tau))d\tau \le 0 \ .$$

When $F(K)$ is bounded, we have chosen T independent of x_0. So we can extend the trajectory $x(\cdot)$ on $[0,T]$ on a trajectory on $[0,2T]$, $[0,3T]$, etc. So, there exists a trajectory defined on $[0,\infty[$. ■

Proof of Proposition 1

Let $x_k(\cdot)$ converging to $x(\cdot)$ uniformly on compact intervals and $v_k(\cdot)$ converging weakly to $v(\cdot)$ in $L^\infty(0,\infty;X)$. Hence

$$(16) \qquad \begin{cases} \text{i)} \ \forall\, t \in [0,\infty[,\ x_k(t) \text{ converges to } x(t) \\ \text{ii)} \ \forall\, T > 0,\ v_k(\cdot) \text{ converges weakly to } v(\cdot) \text{ in } L^1(0,T;X). \end{cases}$$

Actually, it is sufficient to suppose the latter properties (16) hold true. If $\liminf_{k \to \infty} \underset{\sim}{W}(x_k,v_k) = +\infty$, the theorem is true; if not, let

(17) $$c \doteq \liminf_{k \to \infty} \underset{\sim}{W}(x_k,v_k).$$

There exist subsequences (again denoted x_k and v_k) of x_k and v_k such that

(18) $$\forall\, k \in \mathbb{N},\ \underset{\sim}{W}(x_k,v_k) \leq c + \frac{1}{k} .$$

By Mazur's theorem, there exists a sequence of elements

(19) $$w_h(\cdot) = \sum_{k=h}^{\infty} a_h^k v_k(\cdot),\ a_h^k \geq 0 ,\ \sum_{k=h}^{\infty} a_h^k = 1 ,$$

(where $a_h^k = 0$ but for a finite number of indexes k) that converges strongly to $v(\cdot)$ in $L^1(0,T;X)$:

$$\forall\, h \in \mathbb{N},\ \|w_h - v\|_{L^1(0,T;X)} \leq 1/h .$$

Hence, a subsequence (again denoted) w_h converges almost everywhere to $v(\cdot)$. Let $t \geq 0$ be any point where

(20) $$v(t) = \lim_{h \to \infty} w_h(t) .$$

By Proposition 14.2, there exists η such that $\mathcal{E}p\, W(x,\cdot) \subset \mathcal{E}p\, W(x(t),\cdot) + \varepsilon(B\times B)$ when $\|x - x(t)\| \leq \eta$. Let k_0 such that $\|x_k(t) - x(t)\| \leq \eta$ whenever $k \geq k_0$. Since $\mathcal{E}p\, W(x_k(t),\cdot)$ is convex, we deduce that

$$\left(w_h(t), \sum_{k=h}^{\infty} \alpha_h^k W(x_k(t),v_k(t))\right) \in \mathcal{E}p\, W(x_k(t),\cdot) \subset \mathcal{E}p\, W(x(t),\cdot) + \varepsilon(B\times B) .$$

Hence, by letting $h \to \infty$, we obtain

$$\left(v(t), \liminf_{h \to \infty} \sum_{k=h}^{\infty} \alpha_h^k W(x_k(t),v_k(t))\right) \in c\ell(\mathcal{E}p\, W(x(t),\cdot) + \varepsilon(B\times B) .$$

Since this holds true for all $\varepsilon > 0$, and since $\mathcal{E}p\, W(x(t),\cdot)$ is closed, it follows that

(21) $$\left(v(t), \liminf_{h \to \infty} \sum_{k=h}^{\infty} \alpha_h^k W(x_k(t),v_k(t))\right) \in \mathcal{E}p\, W(x(t),\cdot) ,$$

i.e., that

(22) $$\liminf_{h \to \infty} \sum_{k=h}^{\infty} \alpha_h^k W(x_k(t),v_k(t)) \geq W(x(t),v(t)) .$$

We integrate this inequality and we apply Fatou's lemma, which is possible for W is nonnegative. We obtain

$$\underset{\sim}{W}(x,v) \leq \int_0^\infty (\liminf_{h \to \infty} \sum_{k=h}^{\infty} \alpha_h^k W(x_k(t),\, v_k(t))dt$$

$$\leq \liminf_{h \to \infty} \sum_{k=h}^{\infty} \alpha_h^k \underset{\sim}{W}(x_k,v_k) \leq \lim_{h\to\infty} (c + \frac{1}{h}) = c \,.$$

(By (18)). Hence $\underset{\sim}{W}(x,v) \leq c \doteq \liminf_{k \to \infty} \underset{\sim}{W}(x_k,v_k)$. ∎

16. Stability and Asymptotic Stability

We consider the case when $W \equiv 0$; in this particular case, we say that V is a Liapunov function with respect to F if

(1) $\forall x \in K , \quad \exists v \in F(x)$ such that $D_+ V(x)(v) \leq 0$

and that a trajectory $x(\cdot)$ of $x' \in F(x)$ is monotone with respect to V if

(2) the function $t \mapsto V(x(t))$ is nonincreasing.

Hence monotone trajectories remain in the "level sets"

(3) $\{x \in K \mid V(x) \leq V(x_0)\}$.

So, we obtain the following stability property.

Theorem 1

Let K be a closed subset of $\mathbb{R}^n$ and F be a bounded upper semicontinuous map from K to the nonempty compact convex subsets of $\mathbb{R}^n$. Let $V : K \to \mathbb{R}_+$ be a continuous lower semicompact Liapunov function. Let $P_* \doteq \{x \in K \mid V(x_*) = \min_{x \in K} V(x)\}$.
Then the following stability property holds:

For any open neighborhood M of P_* , there exists a neighborhood $N \subset M$ of P_* such that, for all $x_0 \in N$, there exists a trajectory starting at x_0 and remaining in M . ■

Proof.

We set $Q \doteq \{x \in K \mid V(x) \leq \min_{y \in K} V(y) + 1\}$, which is compact. Hence, since M is an open neighborhood of P_* , $Q \cap M$ is compact and $c = \min_{x \in Q \cap \complement M} V(x)$ is finite. Thus the subset $N \doteq \{x \in Q \mid V(x) < c\}$ is contained in M and is a neighborhood of P_* (for V is continuous). Now, if $x_0 \in N$, there exists a trajectory $x(\cdot)$ which is monotone (by Theorem 15.1) and thus, which remains in $N \subset M$ because $V(x(t)) \leq V(x_0) < c$. ■

We shall give now conditions implying asymptotic stability when $W \equiv 0$.

Theorem 2

Let F be an upper semicontinuous map from $K \subset \mathbb{R}^n$ to the compact convex subsets of $\mathbb{R}^n$ and V be the restriction to K of a locally Lipschitzean function $\tilde{V}$ which is lower semicompact. We assume that V is a Liapunov function with respect to F statisying

(4) $\left\{\begin{array}{l} \forall\, x \in K\ ,\ \text{ if there exists } v \in F(x) \text{ such that } D_+ V(x)(v) \geq 0\ , \\ \text{then } V(x) = \min\limits_{y \in K} V(y)\ . \end{array}\right.$

We also assume that

(5) the function $(x,v) \in \text{graph}\ (F) \to D_+ \tilde{V}(x)(v)$ is upper semicontinuous.

(This is the case when, for instance, $\tilde{V}$ is continuously differentiable or convex continuous). Then any monotone trajectory satisfies

(6) $\lim\limits_{t \to \infty} V(x(t)) = \min\limits_{y \in K} V(y)\ .$ ∎

Proof.

We set $v(t) \doteq V(x(t))$. Let us assume that $\alpha \doteq \lim\limits_{t \to \infty} v(t) > \min\limits_{x \in K} V(x) \geq 0$. Let $Q \doteq \{x \in K \mid V(x) \leq V(x_0)\}$, which is compact because V is lower semicompact and lower semicontinuous. Therefore, the graph F_Q of the restriction to Q of F is compact, for F is upper semicontinuous from K to the compact subsets of $\mathbb{R}^n$.

Assumption (4) implies that $D_+ \tilde{V}(x)(v) < 0$ for all $(x,v) \in F_Q$; so, we deduce from Assumption (5) that there exists $\mu > 0$ such that

(7) $\forall\, x \in Q\ ,\ \sup\limits_{v \in F(x)} D_+ \tilde{V}(x)(v) \leq -\mu$

Let $x(\cdot) \in C(0,\infty;\mathbb{R}^n)$ be a trajectory such that $x'(\cdot) \in L^\infty(0,\infty;\mathbb{R}^n)$. Since $x(t) \in Q$ for all $t \geq 0$, we deduce that $D_+ \tilde{V}(x(t))(x'(t)) \leq -\mu$. Also, because V is the restriction of a locally Lipschitzean function $\tilde{V}$, Proposition 6.8 implies that

(8) $\liminf\limits_{h \to 0+} \dfrac{v(t+h) - v(t)}{h} \leq D_+ \tilde{V}(x(t))(x'(t)) \leq -\mu\ .$

Therefore, we deduce from Proposition 6.9 that for all $T = \dfrac{v(0)}{\mu}$, we have

(9) $v(T) - v(0) \leq \displaystyle\int_0^T -\mu\ dt = -v(0)\ .$

Thus $v(T) \leq 0 \leq \min\limits_{x \in K} V(x)$ and $v(T) \geq \alpha > \min\limits_{x \in K} V(x)\ .$

The theorem follows from this contradiction. ∎

17. Liapunov functions for U-monotone maps.

Let $U : K \times K \to \mathbb{R}_+$ be a nonnegative continuous function such that $U(y,y) = 0$ for all $y \in K$, which plays the role of a semidistance (without having to obey the triangle inequality). We shall associate with U the class of "U-monotone" maps F which enjoy the following property. If we know in advance that $x_* \in K$ is a stationary point of F, then the "distance function" to $x_* : x \to U(x,x_*)$ is a Liapunov function. When $U(x,y) \doteq \frac{1}{2} \sum_{i=1}^{r} |x_i - y_i|^2$, the class of U-monotone maps coincides with the class of monotone maps in the usual sense.

We list some examples of functions U.

(1) $$U_p(x,y) \doteq \sum_{i=1}^{n} |x_i - y_i|^p \qquad (1 \leq p < +\infty)$$

and, in particular, for $p = 2$:

(2) $$U_2(x,y) \doteq \frac{1}{2} \sum_{i=1}^{p} |x_i - y_i|^2$$

We mention also

(3) $$U_\infty(x,y) \doteq \max_{i=1,\dots,n} (x_i - y_i)$$

and, if $K = \overset{\circ}{\mathbb{R}}{}^n_+$,

(4) $$U_0(x,y) = \max_{i=1,\dots,n} x_i/y_i - \min_{i=1,\dots,n} x_i/y_i .$$ ■

We associate with U the function U' defined on $K \times K \times X$ by

(5) $$U'(x,y)(v) \doteq D_+(x \mapsto U(x,y))(x)(v) .$$

Now, we introduce the class of U-monotone set-valued maps.

Definition 1

Let $U : K \times K \to \mathbb{R}_+$ be a continuous function such that $U(y,y) = 0$ for all $y \in K$. We say that a set-valued map from K to $\mathbb{R}^n$ is U-monotone if

(6) $$\forall x,y \in K,\ u \in F(x),\ v \in F(y),\ U'(x,y)(v-u) \leq 0 .$$

We say that F is strictly U-monotone if

(7) $$\forall x,y \in K \text{ such that } U(x,y) > 0,\ \forall u \in F(x),\ v \in F(y),\ U'(x,y)(v-u) < 0$$

and strongly U-monotone if there exists $c > 0$ such that

(8) $$\forall x,y \in K,\ u \in F(x),\ v \in F(y),\ U'(x,y)(v-u) + c(U(x,y)) \leq 0$$ ■

Finally, if φ is a bounded continuous map from $\mathbb{R}_+$ to $\mathbb{R}_+$, we say that F is (φ,U)-monotone if

$$(9) \qquad \forall\, x,y \in K\,,\quad u \in F(x),\ v \in F(y),\quad U'(x,y)(v-u) + \varphi(U(x,y)) \leq 0$$

<u>Examples of U-monotone operators</u>

When we take $K = \mathbb{R}^n_+$ and

$$(10) \qquad U_2(x,y) = \frac{1}{2}\sum_{i=1}^{n} |x_i - y_i|^2$$

we see that F is (φ,U_2)-monotone if

$$(11) \qquad \forall\, x,y \in K\,,\quad u \in F(x),\quad v \in F(y),\quad \langle u-v, x-y \rangle \geq \varphi(U_2(x,y))\ .$$

So, usual monotone maps are the U_2-monotone maps.

When $1 < p < \infty$, we set

$$(12) \qquad U_p(x,y) \doteq \frac{1}{p}\sum_{i=1}^{n} |x_i - y_i|^p$$

and

$$(13) \qquad I_0(x,y) \doteq \{i \mid x_i = y_i\}.$$

Therefore, F is (φ,U_p)-monotone if and only if, for all $x,y \in \mathbb{R}^n$, $u \in F(x)$, $v \in F(y)$, we have

$$(14) \qquad \sum_{i \notin I_0(x,y)} |x_i - y_i|^{p-2}(x_i - y_i)(u_i - v_i) \geq \varphi(U_p(x,y))$$

when $p = \infty$, we set

$$(16) \qquad U_\infty(x,y) \doteq \max_{i=1,\dots,n} (x_i - y_i)$$

and

$$(17) \qquad J(x,y) \doteq \{i \mid x_i - y_i = \max_{i=1,\dots,n} (x_i - y_i)\}\ .$$

Then F is (φ,U_∞)-monotone if and only if for all $x,y \in \mathbb{R}^n$, $u \in F(x)$, $v \in F(y)$, we have

$$(18) \qquad \min_{j \in J(x,y)} (u_i - v_i) \geq \varphi(U_\infty(x,y))\ .$$

Finally, if x and $y \in \mathring{\mathbb{R}}^n_+$, we set

$$(19) \qquad U_0(x,y) = \max_{i=1,\dots,n} \frac{x_i}{y_i} - \min_{i=1,\dots,n} \frac{x_i}{y_i}$$

and

(20) $\quad K_+(x,y) = \{i \mid \frac{x_i}{y_i} = \max_i \frac{x_i}{y_i}\}$, $K_-(x,y) = \{i \mid \frac{x_i}{y_i} = \min_i \frac{x_i}{y_i}\}$.

[Note that $U_0(x,y) = 0$ if and only if $\exists \lambda \geq 0$ such that $x = \lambda y$.]

Then F is (φ, U_0)-monotone if and only if for all $x,y \in \mathring{\mathbb{R}}^n_+$, $u \in F(x)$, $v \in F(y)$,

(21) $$\min_{i \in K_+(x,y)} \frac{u_i - v_i}{y_i} - \max_{j \in K_-(x,y)} \frac{u_j - v_j}{y_j} \geq \varphi(U_0(x,y)) \qquad \blacksquare$$

<u>Proof</u>

The above characterizations follow obviously from the computation of the contingent derivatives of $x \to U(x,y)$. Since these functions are convex and continuous, the contingent derivatives coincide with the directional derivatives from the right.

When $1 < p < +\infty$, we set $I_+(x,y) = \{i \mid x_i > y_i\}$ and $I_-(x,y) = \{i \mid y_i < x_i\}$. Then it is clear that

$$U'_p(x,y)(v) = \sum_{i \in I_+(x,y)} (x_i - y_i)^{p-1} v_i - \sum_{i \in I_-(x,y)} (y_i - x_i)^{p-1} v_i$$

$$= \sum_{i \notin I_0(x,y)} |x_i - y_i|^{p-2} (x_i - y_i) v_i \ .$$

If $p = 1$, $U'_1(x,y)(v) = \sum_{i=1}^n v_i$ and, if $p = \infty$, we have $W'(x,y)(v) = \max_{j \in J(x,y)} v_j$. Finally, for $p = 0$, we have $U'_0(x,y)(v) = \max_{i \in K_+(x,y)} \frac{v_i}{y_i} - \min_{i \in K_-(x,y)} \frac{v_i}{y_i}$. $\blacksquare$

The U-monotone maps enjoy the following fundamental properties.

<u>Theorem 1</u>

Let U be a continuous nonnegative function from $K \times K \to \mathbb{R}^n$ such that $U(y,y) = 0$ for all $y \in K$. Let F be a proper bounded upper semicontinuous map from K to the compact convex subsets of $\mathbb{R}^n$. We also posit the following assumptions

(22) $\begin{cases} \text{i) there exists a stationary point } x_* \in K \text{ of } F \\ \text{ii) } -F \text{ is } (\varphi, U) \text{ monotone.} \end{cases}$

Let $w \in C(0,\infty;\mathbb{R})$ be a solution to the differential equation

(23) $\quad w' + \varphi(w) = 0$; $w(0) = U(x_0, x_*)$, x_0 is given in K .

Then there exists a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$, $x(0) = x_0$ such that $t \to U(x(t),x_*)$ is nonincreasing, such that $\int_0^\infty \varphi(U(x(t),x_*))dt < +\infty$ and such that

$$\text{(24)} \qquad \forall\, t \geq 0 \ , \quad U(x(t),x_*) \leq w(t) \ . \qquad \blacksquare$$

Proof.

We set $V(x) \doteq U(x,x_*)$ and $W(x,v) \doteq \varphi(U(x,x_*))$. Since $0 \in F(x_*)$, then, for all $v \in F(x)$, we get $D_+V(x)(v) + W(x,v) = U'(x,x_*)(v - 0) + \varphi(U(x,x_*))$. The right-hand side of this inequality is nonpositive since $-F$ is (φ,U)-monotone. Hence V is a Liapunov function for for F with respect to W. So, we apply successively Theorems 15.1 and 12.3. $\blacksquare$

We mention now a result on asymptotic stability. For simplicity, we prove only a special case.

Theorem 2.

Let $L \subset \mathbb{R}^n$ be an open convex subset of $\mathbb{R}^n$ and $U: L \times L \to \mathbb{R}_+$ be a non-negative continuous function such that

$$\text{(25)} \quad \begin{cases} \text{i)} & \forall\, y \in L \ , \quad U(y,y) = 0 \\ \text{ii)} & \forall\, y \in L \ , \quad x \mapsto U(x,y) \ \text{ is convex.} \end{cases}$$

Let K be a closed subset of L and F be a bounded upper semicontinuous map from K to the compact convex subsets of $\mathbb{R}^n$. We assume also that

$$\text{(26)} \quad \begin{cases} \text{i)} & \text{there exists a stationary point } x_* \in K \text{ of } F \\ \text{ii)} & \forall\, x \in K \ , \quad D_K(x) \cap F(x) \neq \emptyset \\ \text{iii)} & -F \ \text{ is strictly } U\text{-monotone.} \end{cases}$$

Then, for any $x_0 \in K$, there exists a solution to the differential inclusion $x' \in F(x)$, $x(0) = x_0$ satisfying

$$\text{(27)} \qquad \lim_{t \to \infty} U(x(t),x_*) = 0$$

Proof.

We take for Liapunov function the restriction $V|_K$ of the function V defined on L by $V(x) \doteq U(x,x_*)$, where x_* is a stationary point of F. Since V is convex and continuous, then $D_+ V(x)(v) = D_- V(x)(v)$ is upper semicontinuous with

respect to (x,v). It is a Liapunov function since there exists $v \in D_K(x) \cap F(x)$ by assumption (26) ii): Hence $D_+ V|_K(x)(v) \leq D_- V(x)(v) = D_+ V(x)(v) = U'(x,x_*)(v - 0) \leq 0$ because $v \in F(x)$, $0 \in F(x_*)$ and $-F$ is U-monotone. Also, let us assume that there exists $v \in F(x)$ such that $D_+ V(x)(v) \geq 0$. Then $U'(x,x_*)(v - 0) \geq 0$ and, since $v \in F(x)$, $0 \in F(x_*)$ and $-F$ is strictly monotone, we deduce from (7) that $V(x) \doteq U(x,x_*) = 0 = \min_{y \in K} V(y)$. Therefore assumptions of <u>Theorem</u> 16.2 are satisfied and thus, any monotone trajectory statisfies property (27). They do exist by Theorem 15.1. ∎

<u>Remark</u>

We recall that the Browder-Ky Fan theorem states that when K_* is convex and compact, the tangential condition (26) ii) implies the existence of a stationary point $x_* \in K$. ∎

18. Construction of Liapunov functions

If the dynamical system described by the set-valued map $F : K \subset \mathbb{R}^n \to \mathbb{R}^n$ and the function $W:K \times F(K) \to \mathbb{R}_+$ are given, the problem arises whether there exist a Liapunov function V. By Theorems 14.1 and 15.1, we have to find functions V satisfying the property

(1) $\quad \forall x \in K, \quad \exists v \in F(x)$ such that $D_+ V(x)(v) + W(x,v) \le 0$.

For this purpose, we denote by $\mathcal{T}_\infty(x_0)$ the set of viable trajectories of the differential inclusion

(2) $\quad x' \in F(x), \quad x(0) = x_0$ given in K .

We define the function V_F from K to $[0,+\infty]$ by

(3) $$\forall x_0 \in K, \quad V_F(x_0) = \inf_{x(\cdot)\in\mathcal{T}_\infty(x_0)} \int_0^\infty W(x(\tau),x'(\tau))d\tau.$$

We begin by pointing out the following remark.

Proposition 1

Let $V:K \to \mathbb{R}_+$ and $W:K \times F(K) \to \mathbb{R}_+$ be nonnegative functions. If there exists a monotone trajectory $x(\cdot) \in \mathcal{T}_\infty(x_0)$ (with respect to V and W) then

(4) $\quad 0 \le V_F(x_0) \le V(x_0)$. ∎

Proof It follows from Proposition 12.1 and from inequality

$$\forall t \ge 0, \int_0^t W(x(\tau),x'(\tau))d\tau \le V(x_0) - V(x(t)) \le V(x_0). \quad ∎$$

We now prove that V_F Does satisfy the Liapunov condition for F with respect to W .

Proposition 2

Let $K \subset \mathbb{R}^n$ be closed, $F:K \to \mathbb{R}^n$ be a proper upper semicontinuous map with compact convex images and $W:K \times co(F(K)) \to \mathbb{R}_+$ be a nonnegative lower semicontinuous function that is convex with respect to v . If the minimum in $V_F(x_0)$ is achieved for $x_0 \in K$, V_F satisfies the Liapunov condition

(5) $\quad \exists\, v_0 \in F(x_0)$ such that $D_+V_F(x_0)(v_0) + W(x_0,v_0) \le 0$ ∎

Proof.

Let us assume that there exists $x(\cdot) \in \mathcal{T}_\infty(x_0)$ such that $V_F(x_0) = \int_0^\infty W(x(\tau),x'(\tau))d\tau$. Since F is upper semicontinuous, we can associate with any $\varepsilon > 0$ an $\eta > 0$ such that, for all $x \in x_0 + \varepsilon B$, $F(x) \subset F(x_0) + \varepsilon B$. So, for h

small enough, $x'(\tau) \in F(x(\tau)) \subset F(x_0) + \varepsilon B$ for all $\tau \in [0,h]$. On the other hand, $\frac{x(h) - x_0}{h} = \frac{1}{h}\int_0^h x'(\tau)d\tau$ belongs to $F(x_0) + \varepsilon B$ by the mean-value theorem, because the latter set is compact and convex. So, a subsequence $v_n = \frac{x(h_n) - x_0}{h_n}$ converges to some $v_0 \in F(x_0)$. On the other hand,

$$V_F(x(h_n)) \leq \int_{h_n}^{\infty} W(x(\tau),x'(\tau))d\tau = \int_0^{\infty} W(x(\tau),x'(\tau))d\tau$$

$$- \int_0^{h_n} W(x(\tau),x'(\tau))d\tau \leq V_F(x_0) - \int_0^{h_n} W(x(\tau),x'(\tau))d\tau \; .$$

Therefore,

$$\frac{V_F(x_0 + h_n v_n) - V_F(x_0)}{h_n} + \frac{1}{h_n}\int_0^{h_n} W(x(\tau),x'(\tau))d\tau \leq 0 \; .$$

By the very definition of the upper contingent derivative, we have $D_+ V_F(x_0,v_0) = \liminf_{\substack{h \to 0 \\ v \to v_0}} \frac{V_F(x_0 + hv) - V_F(x_0)}{h}$ and, by Proposition 14.1, $W(x_0,v_0) \leq \liminf_{\substack{h_n \to 0 \\ v_n \to v_0}} \frac{1}{h_n}\int_0^{h_n} W(x(\tau),x'(\tau))d\tau$. So, by taking the limit in inequalities (6), we obtain the Liapunov condition (5). ∎

Therefore, by tieing up some preceding results, we can prove that $V_F(\cdot)$ is the smallest Liapunov function with respect to F and W.

We begin by making more precise Theorem 14.1 on necessary conditions.

Theorem 1

Let F be a bounded upper semicontinuous map from $K \subset \mathbb{R}^n$ to the compact convex subsets of $\mathbb{R}^n$. Let W be a nonnegative lower semicontinuous function from $K \times \text{co}F(K)$ which is convex with respect to v.

Let V be a nonnegative function on K. If for all $x_0 \in K$, there exists a monotone trajectory $x(\cdot) \in \mathcal{T}_\infty(x_0)$ with respect to V and W, then not only V is a Liapunov function with respect to F and W, but V_F is also a nonnegative lower semicontinuous Liapunov function smaller than or equal to V. ∎

Proof.

Since there exist monotone trajectories $x(\cdot) \in \mathcal{T}_\infty(x_0)$ with respect to V and W for all $x_0 \in K$, we deduce from Theorem 14.1 that V is a Liapunov function for F with respect to W, from Proposition 1 that $V_F(x_0) \leq V(x_0)$ for all $x_0 \in K$ and from Proposition 2 that V_F is a Liapunov function. The set-valued map $x_0 \to \mathcal{T}_\infty(x_0)$ from K to the space of functions $x(\cdot) \in C(0,\infty;\mathbb{R}^n)$ whose derivatives $x'(\cdot) \in L^\infty(0,\infty;\mathbb{R}^n)$ is upper semicontinuous with compact values when $C(0,\infty;\mathbb{R}^n)$ is supplied with the topology of uniform convergence on compact intervals and when $L^\infty(0,\infty;\mathbb{R}^n)$ is supplied with the weak topology $\sigma(L^\infty,L^1)$. Proposition 15.1 states that the functional $x \mapsto \int_0^\infty W(x(\tau),x'(\tau))d\tau$ is lower semicontinuous on that space. Hence the maximum theorem implies that the function $V_F(\cdot)$ is lower semicontinuous. ■

Another combination of the same arguments yield the following statement:

Theorem 2

Let F be a bounded upper semicontinuous map from a closed subset $K \subset \mathbb{R}^n$ to the compact convex subsets of $\mathbb{R}^n$, satisfying the tangential condition

(7) $\quad \forall x \in K, \; F(x) \cap D_K(x) \neq \emptyset$.

Let $W : K \times \text{co}F(K) \to \mathbb{R}_+$ be a nonnegative lower semicontinuous function which is convex with respect to v. If for $x_0 \in K$, $V_F(x_0)$ is finite, then $V_F(x_0)$ is lower semicontinuous at x_0 and satisfies

(8) $\quad \exists v_0 \in F(x_0)$ such that $D_+ V_F(x_0) + W(x_0,v_0) \leq 0$.

Consequently, if $V_F(\cdot)$ is finite on K, it is a lower semicontinuous Liapunov function for F with respect to W. ■

The fact that V_F is a Liapunov function yields the following characterization of trajectories achieving the minimum of V_F.

Proposition 3.

Let us assume that the function V_F defined by (3) is finite on K and is a Liapunov function for F with respect to W. Then the trajectories of the differential inclusion $x' \in F(x)$, $x(0) = x_0$, which are monotone with respect to V_F and W achieve the minimum in $V_F(x_0)$. ■

Proof.

If $x(\cdot)$ is a monotone trajectory with respect to V_F and W, we obtain the inequality

(9) $$\int_0^\infty W(x(\tau),x'(\tau))d\tau \leq V_F(x_0) = \inf_{y(\cdot)\in\mathcal{T}_\infty(x_0)} \int_0^\infty W(y(\tau),y'(\tau))d\tau. \quad \blacksquare$$

This simple statement is a very important consequence in control theory. We associate with V_F the new map G defined by

(10) $$G(x) \doteq \{v \in F(x) \mid D_+ V_F(x)(v) + W(x,v) = \min_{w\in F(x)} (D_+ V_F(x)(w) + W(x,w))\}.$$

Note that $G(x)$ is single-valued when $v \to D_+ V_F(x)(v)$ is convex and $v \mapsto W(x,v)$ is strictly convex for all $x \in K$. One can devise sufficient conditions implying that $(x,v) \to D_+ V_F(x)(v)$ is upper semicontinuous. This and the continuity of W guarantees that G is an upper semicontinuous map.

In any case, by Theorem 16.1, solutions $x(\cdot)$ of the differential inclusion $x' \in G(x)$, $x(0) = x_0$ yield trajectories of $x' \in F(x)$, $x(0) = x_0$ which achieve the minimum in $V_F(x_0)$. $\blacksquare$

19. Construction of dynamical systems having monotone trajectories

The question arises whether V and W being given, there exists a continuous single-valued map f such that the differential equation $x' = f(x)$ has monotone trajectory with respect to V and W.

In this section, we shall assume that

$$(1)\quad \begin{cases} \text{i) } K \text{ is compact and convex} \\ \text{ii) } V \text{ is the restriction to } K \text{ of a convex continuous function } \tilde{V} \\ \text{iii) } W \text{ is continuous and convex with respect to } v . \end{cases}$$

We recall that a necessary and sufficient condition for f to have monotone trajectories with respect to V and W is that

$$(2)\qquad \forall x \in K ,\quad D_+ V(x)(f(x)) + W(x,f(x)) \leq 0 .$$

Since $D_+ V(x)(v)$ is the restriction to the tangent cone $D_K(x)$ of $D_+ \tilde{V}(x)(v)$, we set

$$(3)\qquad S(x) \doteq \{v \in X \mid D_+ \tilde{V}(x)(v) + W(x,v) \leq 0 \} .$$

So, the necessary and sufficient conditions can be written

$$(4)\qquad \forall x \in K ,\quad f(x) \in S(x) \cap D_K(x) .$$

In order to exclude the obvious solution $f \equiv 0$, we introduce the cones

$$(5)\qquad \overset{\circ}{S}(x) = \{v \in X \mid D_+\tilde{V}(x)(v) + W(x,v) < 0\}.$$

which may be empty. We also set

$$(6)\qquad K_0 \doteq \{x \in K \mid \overset{\circ}{S}(x) \cap D_K(x) \neq \emptyset \};\quad K_1 \doteq K \cap \complement K_0 .$$

<u>Theorem</u> (Cornet)

Let K be a compact convex subset, V be the restriction to K of a continuous convex function $\tilde{V}$ and W be a nonnegative continuous function on $K \times \mathbb{R}^n$ which is convex with respect to v. We assume that the subset K_0 defined by (6) is nonempty.

Then there exists a continuous function f whose set of stationary points is K_1 such that the differential equation $x' = f(x)$, $x(0) = x_0$ has a monotone trajectory with respect to V and W. ∎

Proof.

Since $(x,v) \mapsto D_+ \tilde{V}(x)(v)$ is upper semicontinuous, the graph of the map $\mathring{S}$, which is equal to

$$\text{Graph } \mathring{S} = \{(x,v) \in K_0 \times \mathbb{R}^n \mid D_+ V(x)(v) + W(x,v) < 0\}$$

is open. Since the set-valued map $x \to D_K(x)$ is l.s.c. and has convex values $x \to \mathring{S}(x) \cap D_K(x)$ is locally selectionable from K_0 to $\mathbb{R}^n$ and its images are convex cones. (See Cornet [1]). Hence, by results of Corret [1], there exists a continuous function f from K ot $\mathbb{R}^n$ satisfying

(7) $\forall x \in K_0$, $f(x) \in D_K(x) \cap \mathring{S}(x)$ and $\forall x \in K_1$, $f(x) = 0$.

So, such a function f satisfies the assumptions of Theorem 15.1. Hence there exists monotone trajectories of the differential equation $x' = f(x)$. ■

20. Feedback controls yielding monotone trajectories.

Let U be a set of controls u and $f : K \times U \to X$ the map that assigns to each state x and to each control u the velocity $f(x,u)$ of the state.

A feedback control is a map $\underset{\sim}{u} : x \in K \to \underset{\sim}{u}(x) \in U$ associating with each state of the system a control according to a fixed rule for achieving a given purpose.

The example of such a purpose is the requirement that the trajectories of the differential equation

$$(1) \quad \begin{cases} \text{i)} & x'(t) = f(x(t), \underset{\sim}{u}(x(t))) \\ \text{ii)} & x(0) = 0 \end{cases}$$

exist and satisfy the monotonicity condition:

$$(2) \quad \forall\, s > t\,, \quad V(x(s)) - V(x(t)) + \int_t^s W(x(\tau), x'(\tau))\,d\tau \leq 0\,.$$

We assume that V is the restriction to K of a convex continuous function $\tilde{V}$.

We introduce the following set-valued map $\overset{\circ}{S}$ defined by

$$(3) \quad \overset{\circ}{S}(x) \doteq \{v \in X \mid D_+ \tilde{V}(x)(v) + W(x,v) < 0\}$$

and we set

$$(4) \quad K_0 = \{x \in K \mid \overset{\circ}{S}(x) \cap D_K(x) \neq \emptyset\}, \quad K_1 = K \cap \complement K_0\,.$$

<u>Theorem 1</u>

Let us assume that $K \subset X$ and U are both convex compact subsets, that U contains 0 and that $F : K \times U \to X$ is a continuous map that is linear with respect to the controls. We assume that $K_0 \neq \emptyset$ and that there exists $\gamma > 0$ such that

$$(5) \quad \begin{cases} \forall\, x \in K_0\,, \ \forall\, y \in X\,, \ \|y\| \leq \gamma\,, \ \exists\, u \in U \text{ such that} \\ \qquad f(x,u) - y \in D_K(x) \cap \overset{\circ}{S}(x)\,. \end{cases}$$

Then there exists a feedback control $\underset{\sim}{u} \in C(K,U)$, vanishing on K_1 that provides monotone trajectories with respect to V and W. ■

<u>Proof</u>.

By <u>Theorem</u> 15.1, we have to prove the existence of a feedback control $\underset{\sim}{u} \in C(K,U)$ such that

$$(6) \quad \begin{cases} \text{i)} & \forall\, x \in K_0\,, \ f(x, \underset{\sim}{u}(x)) \in D_K(x) \cap \overset{\circ}{S}(x) \\ \text{ii)} & \forall\, x \in K_1\,, \ \underset{\sim}{u}(x) = 0\,. \end{cases}$$

Let us set, if $x \in K_0$,

(7) $$G(x) \doteq \{u \in U \mid f(x,u) \in D_K(x) \cap \mathring{S}(x)\}.$$

We already mentioned that $x \mapsto D_K(x) \cap \mathring{S}(x)$ is locally selectionable and thus, lower semicontinuous. Assumption (5) **implies** that G is lower semicontinuous on K_0. (See Aubin and Cellina [1]).

Michael's theorem states that there exists a continuous selection $\underset{\sim}{v} \in C(K_0,U)$ of the set-valued map C. We denote by $d_{K_1}(x)$ the distance from x to K_1 and we set:

(8) $$\underset{\sim}{u}(x) = \begin{cases} d_{K_1}(x)\, \underset{\sim}{v}(x) & \text{if } x \in K_0 \\ 0 & \text{if } x \in K_1 . \end{cases}$$

This function $\underset{\sim}{u}$ is continuous on K. It is obviously true when $x \in K_0$. Let us check that it is continuous at $x \in K_1$. Let $\varepsilon > 0$ and $y \in x + \varepsilon B$. Then $\underset{\sim}{u}(x) = 0$ and either $\underset{\sim}{u}(y) = 0$ (when $y \in K_1$) or $\underset{\sim}{u}(y) \leq d_{K_1}(y)M \leq \varepsilon$ where $M = \|U\|$. Then $\|\underset{\sim}{u}(x) - \underset{\sim}{u}(y)\| = \|\underset{\sim}{u}(y)\| \leq \varepsilon M$. Hence $\underset{\sim}{u}$ is continuous. Since $\underset{\sim}{v}(x) \in G(x)$ and since f is linear with respect to the controls, we deduce that

$$f(x,\underset{\sim}{u}(x)) = d_{K_1}(x)\, f(x,\underset{\sim}{v}(x)) \in D_K(x) \cap \mathring{S}(x)$$

when $x \in K_0$ and that $f(x,\underset{\sim}{u}(x)) = 0$ when $x \in K_1$. ∎

21. The time dependent case.

We shall adapt to the time dependent case the results we proved for the time independent case. We only have to use the classical transformation which amounts to observing that the solutions to the differential inclusion

(1) $\quad x'(t) \in F(t,x) \; ; \quad x(t_0) = x_0$

and the solutions $\tau \mapsto (t(\tau),x(\tau)) \doteq \hat{x}(\tau)$ of the differential inclusion

(2) $\quad \hat{x}' \in \hat{F}(\hat{x}) \; , \quad x(0) = (t_0,x_0)$

where we set

(3) $\quad \forall \; (t,x) \doteq \hat{x} \in \mathrm{Dom}(F) \qquad \hat{F}(\hat{x}) \doteq (1,F(t,x))$.

We shall denote by $\hat{K} \doteq \mathrm{Dom}\, F$ the domain of F , which is the domain of $\hat{K}$.

We introduce

$$(4) \quad \begin{cases} \text{i) a nonnegative function } V \text{ from } \hat{K} \text{ to } \mathbb{R} \\ \text{ii) a nonnegative function } W \text{ from } \hat{K} \times \mathrm{co}F(\hat{K}) \text{ to } \mathbb{R} . \end{cases}$$

Now, the symbol $D_+ V(t,x)(1,v)$ denotes the upper contingent derivative of V at (t,x) in the direction $(1,v)$. We recall that when V is differentiable, we have

$$(5) \quad D_+ V(t,x)(1,v) = \frac{\partial}{\partial t} V(t,x) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} V(t,x) v_i \; .$$

<u>Proposition</u> 1

$\hat{x}(\cdot)$ is a monotone trajectory of the differential inclusion (2) with respect to V and W if and only if $x(\cdot)$ is a trajectory of the differential inclusion (1) which is monotone with respect to V and W in the sense that

$$(6) \quad \forall \; s \geq t \; , \quad V(s,x(s)) - V(t,x(t)) + \int_0^\infty W(\tau,x(\tau),x'(\tau))d\tau \leq 0 \; .$$

<u>Proof</u>.

Indeed, $\hat{x}(\cdot)$ is a solution to (2) if and only if $\hat{x}(t) = (t,x(t))$ where $x(\cdot)$ is a solution to (1). Note that $\hat{x}'(t) = (1,x'(t))$. So, Condition (6) is equivalent to $\forall \; s \geq t \; , \quad V(\hat{x}(s)) - V(\hat{x}(t)) + \int_0^\infty W(\hat{x}(\tau),\hat{x}'(\tau))d\tau \leq 0$. ∎

We introduce now the concept of Liapunov function.

Definition 1

Let F be a set-valued map from $\tilde{K} \subset \mathbb{R}_+ \times \mathbb{R}^n$ to $\mathbb{R}^n$, V be a nonnegative function from $\tilde{K}$ to $\mathbb{R}$ and W be a nonnegative function from $\tilde{K} \times co(F(K))$ to $\mathbb{R}$. We say that V is a Liapunov function for F with respect to W if

(7) $\quad \forall (t,x) \in \tilde{K}, \quad \exists v \in F(t,x)$ such that $D_+ V(t,x)(1,v) + W(t,x,v) \leq 0$. ■

We can consider $\tilde{K} \subset \mathbb{R} \times \mathbb{R}^n$ as the graph of a set-valued map $t \to K(t)$. Then monotonicity condition (6) implies in particular that

(8) $\quad \forall t, \quad x(t) \in K(t))$

and the Liapunov condition (8) implies in particular that

(9) $\quad \forall t \geq 0, \quad \forall x \in K(t), \quad v \in D K(t,x)(1)$

since the latter condition is equivalent to the tangential condition $(1,v) \in D_{\tilde{K}}(t,x)$. When V is the restriction to $\tilde{K}$ of a continuous convex function $\tilde{V}$, the Liapunov condition can be written

$$(10) \quad \begin{cases} \forall t \geq 0, \quad \forall x \in K(t), \quad \exists v \in F(t,x) \cap D K(t,x)(1) \text{ such that} \\ D_+ \tilde{V}(t,x)(1,v) + W(t,x,v) \leq 0 . \end{cases}$$

When V is the restriction to $\tilde{K}$ of a continuously differentiable function $\tilde{V}$, the Liapunov condition can be written

$$(11) \quad \begin{cases} \forall t \geq 0, \quad \forall x \in K(t), \quad \exists v \in F(t,x) \cap D K(t,x)(1) \text{ such that} \\ \frac{\partial}{\partial t} V(t,x) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} V(t,x) v_i + W(t,x,v) \leq 0 . \end{cases}$$

We deduce from Theorems 14.1 and 15.1 the following characterization of existence of monotone trajectories.

Theorem 1

Let F be a bounded upper semicontinuous map from $\tilde{K} \subset \mathbb{R}_+ \times \mathbb{R}^n$ to the compact convex subsets of $\mathbb{R}^N$, V be a nonnegative continuous function from $\tilde{K}$ to $\mathbb{R}$ and W be a nonnegative continuous function from $K \times coF(K)$ to $\mathbb{R}$ which is convex with respect to the last argument. Then the differential inclusion

(1) $\quad x'(t) \in F(t,x(t)) \; ; \quad x(t_0) = x_0$

has a monotone trajectory $x(\cdot) \in \mathcal{C}(t_0,\infty;\mathbb{R}^n)$ for all $t_0 \geq 0$ and $x_0 \in K(t_0)$ if and only if V is a Liapunov for F with respect to W. ∎

We mention also the following adaptation of Theorem 18.2.

We denote by $\mathcal{T}_\infty(t_0,x_0)$ the set of solutions $x(\cdot) \in \mathcal{C}(t_0,\infty;x)$ of the differential inclusion (1). We introduce the function

$$V_F(t_0,x_0) = \inf_{x(\cdot)\in\mathcal{T}_\infty(t_0,x_0)} \int_0^\infty W(\tau,x(\tau),x'(\tau))d\tau. \tag{12}$$

Theorem 2

Let F be a bounded upper semicontinuous map from a closed graph of a set-valued map $K(\cdot):\mathbb{R}_+ \to \mathbb{R}^n$ to the compact convex subsets of $\mathbb{R}^n$, satisfying

$$\forall\, t \geq 0\,,\ \forall\, x \in K(t),\ F(t,x) \cap D\,K(t,x)(1) \neq \emptyset\ . \tag{13}$$

Let $W : \hat{K} \times \overline{co}(F(\hat{K})) \to \mathbb{R}_+$ be a nonnegative lower semicontinuous function which is convex with respect to the last argument. If for all $(t_0,x_0) \in \hat{K}$, the function $V_F(t_0,x_0)$ is finite, it is the smallest nonnegative lower semicontinuous Liapunov function for F with respect to W. ∎

If $V(t_0,x_0)$ is a liapunov function for F with respect to W, then

$$\inf_{v\in F(t,x)} [D_+ V(t,x)(1,v) + W(t,x,v)] \leq 0\ . \tag{14}$$

When V is the restriction to $\hat{K}$ of a differentiable function $\tilde{V}$, this inequality can be written

$$\frac{\partial V}{\partial t}(t,x) + \inf_{v\in F(t,x)\cap DK(t,x)(1)} \Big[\sum_{i=1}^n \frac{\partial V}{\partial x_i}(t,x)v_i + W(t,x,v)\Big] \leq 0\ . \tag{15}$$

This is the Carathéodory-Hamilton-Jacobi-Bellman equation of control theory. ∎

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We use the Bouligand contingent cone to a subset K of a Hilbert space at $x \in K$ for defining contingent derivatives of a set-valued map, whose graphs are the contingent cones to the graph of this map, as well as the upper contingent derivatives of a real valued function. We develop a calculus of these concepts and show how they are involved in optimization problems and in solving equations $f(x)=0$ and/or inclusions $0 \in F(x)$. They also play a fundamental role for generalizing the Nagumo theorem on flow invariance and for generalizing the concept of Liapunov functions for differential equations and/or differential inclusions.

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